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ABSTRACT

This document has been prepared to help teachers and administrators in New York plan and execute a mathematics program for grade eight. The syllabus is designed to improve the problem solving abilities of students, while continuing a natural development of mathematical skills and concepts. Problem solving situations are emphasized in each of seven major strands of study. These are: (1) an introduction to problem solving; (2) number concepts; (3) ratio, proportion, and percent; (4) probability and statistics; (5) algebra and coordinate geometry; (6) computer awareness; and (7) geometry and measurement. Within each section is some background information for the teacher, along with teaching strategies, example problems, and their answers. Also included in the syllabus is a content outline and suggested time allotments, along with suggestions for modifying instructional techniques and materials for students with handicapped conditions. (TW)

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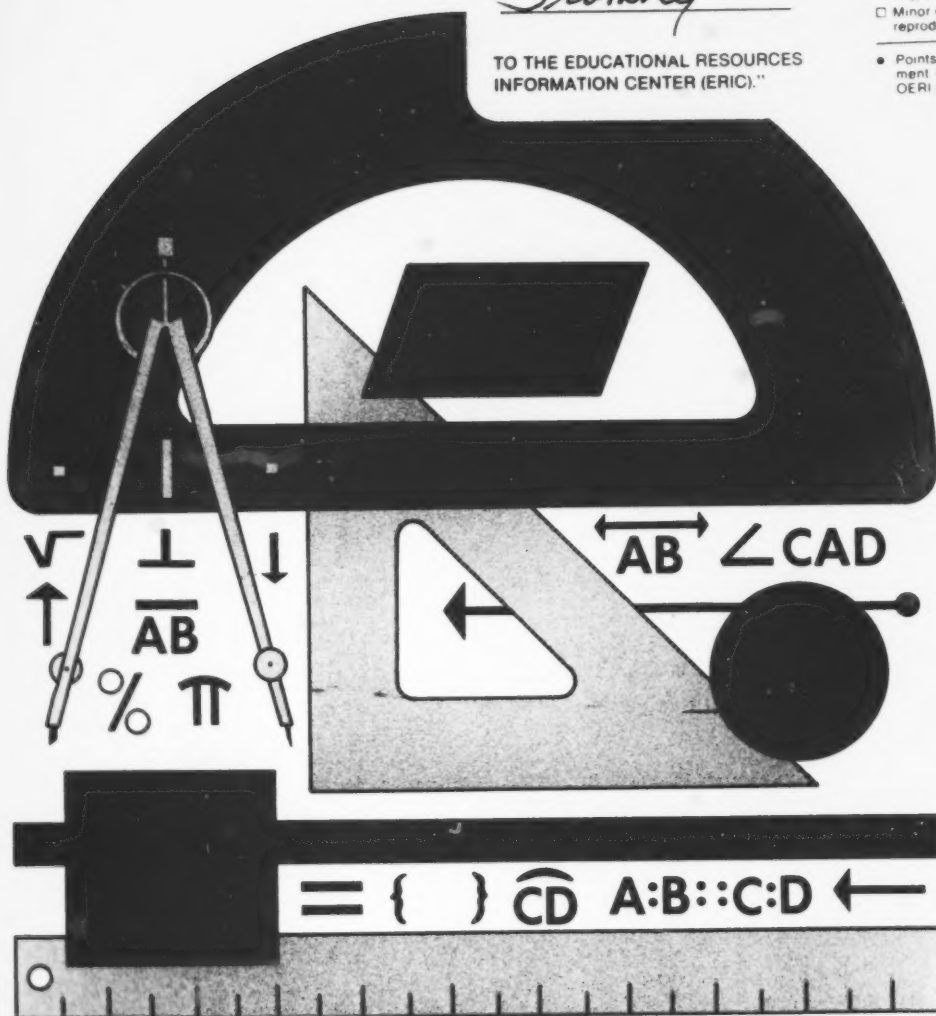
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MATHEMATICS

8 SYLLABUS



The University of the State of New York / THE STATE EDUCATION DEPARTMENT
Bureau of Mathematics Education / Bureau of Curriculum Development
Albany, New York 12234

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TO: Persons with Responsibilities for Implementing Mathematics Programs

FROM: Edward T. Lalor, Director, Division for Program Development

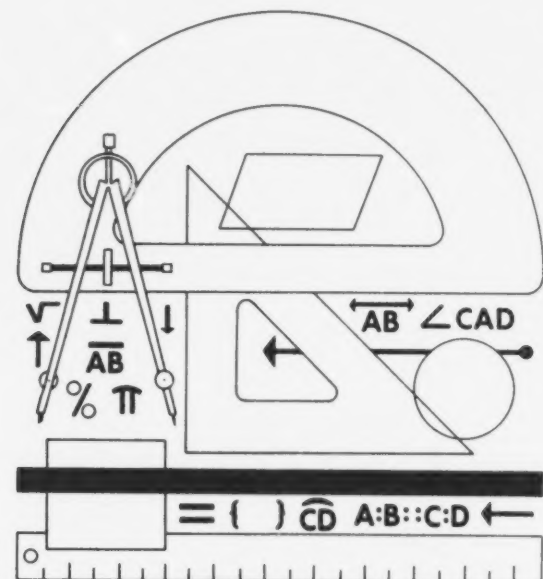
 Charles J. Trupia, Director, Division of General Education

This publication, *Mathematics 8*, is the syllabus designed to continue the developmental mathematics program established by *Mathematics 7*. It is intended to serve as the basis for local activities such as:

- Development of local courses of study to meet the standards stated in the syllabus
- Selection and acquisition of support materials
- Articulation of local offerings from grade to grade or building to building
- Evaluation of student performance

As is the policy of the State Education Department, this syllabus will be periodically reviewed. Your comments on it are welcome and should be addressed to:

Bureau of Mathematics Education
New York State Education Department
Albany, NY 12234



MATHEMATICS 8

Syllabus

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The University of the State of New York
THE STATE EDUCATION DEPARTMENT

Bureau of Mathematics Education
Bureau of Curriculum Development
Albany, New York 12234

1988

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FOREWORD

This document has been prepared to help teachers and administrators plan and execute a mathematics program for grade 8.

In both content and approach, the strands reflect the guidelines and philosophy formulated by an advisory committee of mathematics teachers assembled in the spring of 1981:

Rose Ambrosino, Smithtown
William Collins, Syracuse
Joyce Evans, Afton
Herbert Fremont, Queens College
Eileen Gress, Fairport
Marvin Itzkowitz, New York City
Norman Michaels, Rochester
Susanne Reeder, Buffalo

Two of the committee members, the late Herbert Fremont and Eileen Gress, were selected as writers. By January of 1982, a content outline and the beginnings of a first draft were ready for review. Teachers then engaged in teaching mathematics in grade 8 were asked to read the material and submit their comments in writing. These were compiled during the summer of 1982. In November 1982 a new writer, Edward Keenan of the East Williston Schools, was commissioned to write a *Syllabus* for mathematics in grade 8 that:

- Was based on the guidelines and philosophy of the advisory committee,
- Incorporated aspects of the material already developed and the suggestions of the reviewers, and
- Was suitable for fieldtesting.

In the spring of 1983, 41 pilot districts were selected from over 100 volunteers. Care was taken to choose schools that reflected all types of districts (urban, rural, and suburban) in all regions of the State. The districts included:

Attica
Auburn
Ballston Spa
Bay Shore
Binghamton (parochial)
Buffalo
Burnt Hills
Carthage
Cheektowaga
Clinton
Cobleskill
Division for Youth
Dobbs Ferry
Duanesburg
East Greenbush

East Syracuse
Fairport
Geneva
Hicksville
Iroquois
Kenmore
Madrid-Waddington
Manhasset
Mastic Beach
Miller Place
Mt. Vernon
New Hartford
New York City
Newburgh
Norwood-Norfolk

Orchard Park
Pelham
Riverhead
Schenectady
Seneca Falls
Utica

Vestal
Wappingers Falls
Watervliet
West Irondequoit
Whitesboro

In June 1986, after two years of pilot testing, these schools evaluated the document and sent their comments to the Bureau of Mathematics Education. These evaluations were compiled and analyzed by Norman Swanson of Hudson Valley Community College. During July 1986, Joseph Gehringer of the New York City School District #2 and Edward Keenan revised the draft material to reflect the concerns expressed in the evaluations.

This *Syllabus* for mathematics in grade 8 was prepared under the direction of Dorothy Foley, Chief, Bureau of Curriculum Development, and Fredric Paul, Chief, Bureau of Mathematics Education. Lynn A. Richbart and Toni Meyer, Associates, Bureau of Mathematics Education, coordinated the field testing and evaluation of the manuscript. Staff of the Office of Education of Children with Handicapping Conditions reviewed the manuscript in terms of the needs of students with handicapping conditions. Michael Moon, Associate, Division of Civil Rights and Intercultural Relations, reviewed the manuscript for equity concerns. Rita Sator and Robert Carruthers, Associates, Bureau of Curriculum Development, prepared its format and edited the manuscript.

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INTRODUCTION

A. *Content of the Syllabus*

This *Syllabus* for grade 8 is designed to improve the problem-solving abilities of the students, while continuing a natural development in areas such as mathematical skills and concepts. To foster this approach, problem-solving situations are an integral part of each strand of study.

In extending the pattern established in *Mathematics 7 Syllabus*, the *Syllabus* for grade 8 is organized into seven major strands of study:

- I. An Introduction to Problem Solving
- II. Number Concepts
- III. Ratio, Proportion, and Percent
- IV. Probability and Statistics
- V. Algebra and Coordinate Geometry
- VI. Computer Awareness
- VII. Geometry and Measurement

B. *Instructional Considerations*

Although the strands are listed separately, it is expected that there will be many points of intersection between them resulting in a unified program in mathematics for grade 8, through the development of activities that grow out of the following fundamental principles:

- Problem solving is the basic approach to the program and is utilized in every strand.
- After studying basic operations with whole numbers, common fractions, and decimals in the Number Concepts strand, the students should apply these skills as an integral part of problem-solving activities in each strand.
- Integration of topics should occur naturally and not be forced.
- Mathematics is to be explored at an informal level; wherever possible, formal operations and concepts are to be developed through the use of concrete models.
- Alternative algorithms are presented, and their use is encouraged.
- Although the students still must demonstrate competence in mathematical operations, computers and hand-held calculators are valuable tools that may be used in developing concepts, in finding patterns, in checking work, and (as in statistics) in removing some of the computational drudgery from mathematical problem solving. The use of computers and hand-held calculators may be especially important for those students who have the ability to acquire the concepts and skills presented in the *Syllabus* but who, because of a handicapping condition, have difficulty in demonstrating competence in computation. Some students with handicapping conditions may need to use alternative types of calculators such as those which are voice-activated, braille, or those which have an enlarged keyboard.

C. *Use of the Syllabus*

It is recommended that the study of mathematics in grade 8 begin with problem solving. The order in which the other strands are presented in the *Syllabus* is not intended to indicate the order in which the topics must be studied. Teachers should feel free to develop these mathematical ideas in a sequence that seems best to them.

In the material which follows, several topics are optional and should be included only as time and student ability permit.

A suggested time allotment is given for each major unit and is intended as a general guide in planning the entire year's work.

Suggested instructional strategies are included for each topic within the outline. These strategies are intended to give teachers ideas, methodologies, and examples of what might be done within a particular topic. Teachers are encouraged to adapt these strategies to their own particular style of teaching and to the individual needs of the students.

D. Strategies for Modifying Instructional Techniques and Materials for Students with Handicapping Conditions

Students who are educationally handicapped have been classified by the local school district Committee on Special Education (CSE) and receive special education to the extent needed for progress in school. Most students who require special education attend regular classes in conjunction with specialized instruction and/or related services.

The majority of students with handicapping conditions have, by definition, the intellectual capacity to master the curricular content requirements for a high school diploma. Such students must attain the same academic standards as their nonhandicapped peers in order to meet these requirements. Students with handicapping conditions are provided instruction in a wide variety of settings from regular education to special education classes.

Teachers providing instruction through this *Syllabus* should become aware of the needs of those students with handicapping conditions who have been appropriately placed within their classes. Instructional techniques and materials must be modified if needed so that the information can be attained by such students. Teachers in both regular and special education will need to maintain ongoing communication on programmatic and individual student needs.

1. Students with handicapping conditions may use alternative testing techniques. The needed testing modifications must be identified in the student's Individualized Education Program. Both special and regular education need to work in close cooperation so that the testing modifications can be used consistently throughout the student's program.
2. The use of computer software may be appropriate for activities that require a significant amount of writing by students.
3. Identify, define and pre-teach key vocabulary. Many terms in this *Syllabus* are specific and may need continuous reinforcement for some students with handicapping conditions. It would also be helpful to provide a list of these key words to the special education teacher in order to provide additional reinforcement in the special education setting.
4. Assign a partner for the student as an additional resource to facilitate clarification of daily assignments, timelines for assignments, and access to daily class notes.
5. To increase accessibility of computers and software to students with handicapping conditions, modifications such as eye latches, voice input synthesizers, oversized key pads and loading software and other types of input and output devices should be considered.

GOALS AND OBJECTIVES

The Regents Statement of Goals for students includes the following:

1. Each student will master computation skills as a foundation to
 - think logically and creatively
 - apply reasoning skills to issues and problems
 - perform basic mathematical calculations
 - determine what information is needed for particular purposes and be able to acquire, organize, and use that information for those purposes.
2. Each student will learn methods of inquiry and knowledge through mathematics and use the methods and knowledge in interdisciplinary applications.

This *Syllabus* uses these Regents Goals as a cornerstone. Its objectives support the National Council of Teachers of Mathematics' *An Agenda for Action*, which recommends that problem solving be the focus of school mathematics and that basic skills in mathematics be defined to encompass more than computational facility.

The National Council of Supervisors of Mathematics' *Basic Mathematical Skills* defines ten such areas:

1. Problem Solving
2. Applying Mathematics to Everyday Situations
3. Alertness to the Reasonableness of Results
4. Estimation and Approximation
5. Appropriate Computational Skills
6. Geometry
7. Measurement
8. Reading, Interpreting, and Constructing Tables, Charts, and Graphs
9. Using Mathematics to Predict
10. Computer Literacy

Specific content objectives relating to each of these basic skills are listed in the Content Outline of this *Syllabus* and are described in its Suggestions to Teachers.

CONTENT OUTLINE AND SUGGESTED TIME ALLOTMENTS

Unit Topic	Time(Days)
I. An Introduction to Problem Solving	3-4
A. Review General Technique	
B. Problem-Solving Strategies, and Extending a Problem by:	
1. Altering the problem or the question	
2. Seeking a general solution	
3. Noting restrictions to the general solution	
4. Studying cases where the general solution does not apply	
II. Number Concepts	35-38
A. Whole Numbers	
1. Sequences and patterns	
2. Powers of whole numbers	
a. Positive integral and zero exponents	
b. Operations of multiplication, division, and raising to a power	
B. Integers	
1. Operations	
a. Absolute value	
b. Sequences with integral terms	
c. Problem solving	
2. Properties involving operations with integers	
a. Commutative	
b. Associative	
c. Distributive	
d. Identity elements for addition and multiplication	
e. Inverse element for addition	
f. Closure (<i>optional</i>)	
3. Roots	
a. Two square roots of a positive number	
b. Principal square root	
c. Principal cube root (<i>optional</i>)	
4. Order of operations, including roots	
C. Rational Numbers	
1. Operations and problem solving	
a. Positive and negative common fractions	
b. Positive and negative decimal fractions	
2. Properties involving operations with rational numbers	
3. Powers and roots of rational numbers	
a. Powers with positive integral and zero exponents	
b. Square roots that are rational numbers	
4. Negative integral exponents, with natural number bases	

5. Scientific notation
 - a. Numbers larger than or equal to 1
 - b. Numbers less than 1 but greater than 0
 - c. Operations involving numbers in scientific notation: multiplication and division (*optional*)
 6. Rational numbers defined
- D. Irrational Numbers
1. Definition and examples
 2. Rational approximation and square root
 - a. Trial and error
 - b. Divide and average
- E. Real Numbers
1. Definition and subsets of the real numbers
 2. The real number line
- III. Ratio, Proportion, and Percent 22-25
- A. Ratio and Proportion
1. Ratio and rate
 2. Proportions and comparisons
 - a. Cross-products rule and equations
 - b. Order relations and inequalities
 3. Problem solving with proportions
 - a. Consumer problems
 - b. Rates, including those related to distance and time
 - c. Scale drawing
 - d. Similar polygons, especially triangles
- B. Right Triangle Trigonometry
1. Indirect measure
 2. Naming sides in a right triangle
 3. Trigonometric ratios
 - a. Tangent
 - b. Sine
 - c. Cosine
- C. Percents
1. Percents greater than 100% and less than 1%
 2. Working with percents
 - a. Finding a percent of a number
 - b. Finding what percent one number is of another
 - c. Finding a number when a percent of it is given
 3. Problem solving
 - a. Percent of increase and percent of decrease
 - b. Compound interest
 - c. Applications from print media
 - d. Circle graphs
- IV. Probability and Statistics 20-23
- A. Probability
1. Review of theoretical probability

2. Counting procedures related to probability
 - a. Independent events
 - b. Extending the Counting Principle
 3. Permutations
 - a. Definition
 - b. Factorial notation
 - c. n objects taken n at a time
 - d. n objects taken r at a time
 4. Probability of two or more events
 - a. Sample space for two or more independent events
 - b. Probability of two or more independent events
 - c. Probability of two mutually exclusive events
 - d. Applications
- B. Statistics
1. Grouped data using intervals of length one
 - a. The mean
 - b. The median
 - c. The mode
 2. Presentation of organized data
 - a. Histogram
 - b. Frequency polygon
- V. Algebra and Coordinate Geometry 35-38
- A. Algebraic Expressions and Sentences
1. Forming algebraic expressions and sentences
 - a. Review of terminology
 - b. Translating verbal phrases
 2. Evaluating algebraic expressions
 3. Translating verbal sentences into formulas
- B. Solving First-Degree Equations and Inequalities in One Variable
1. Properties of equality
 2. A plan for solving problems algebraically
 - a. General technique
 - b. Equations
 - c. Inequalities
 3. Solutions of open sentences involving two or more steps
 - a. Like terms
 - b. Parentheses
- C. Equations and the Coordinate Plane
1. Functions of the form $y = ax$, through applications
 - a. Table of values
 - b. Sentence
 - c. Formula
 - d. Coordinate graph
 - e. Predictions
 2. Functions of the form $y = ax + b$, through applications
 3. Coordinate graphs of $x = a$ and $y = b$

- D. Inequalities and the Coordinate Plane
 - 1. Number line graphs of inequalities
 - 2. Coordinate graphs of inequalities (*optional*)
 - 3. Applications of inequalities

VI. Computer Awareness

5-11

- A. Introduction to Computers
 - 1. Components of a modern-day digital computer (*optional*)
 - 2. Careers (*optional*)
 - 3. The loop as a computer process
 - a. Flowcharts
 - b. Programs (*optional*)
- B. Mathematical Operations, Functions, and Relations
 - 1. Operations using computer symbols
 - 2. Evaluating expressions in computer format
 - a. Order of operations
 - b. Use of decimals in output
 - c. Use of scientific notation in output
 - 3. Functions used with computers
 - a. Absolute value function, ABS (X)
 - b. Greatest integer function, INT (X)
 - c. Square root function, SQR (X)
 - d. Random number function, RND (1)
 - 4. Relations used with computers (*optional*)
 - a. Simple inequalities
 - b. Compound inequalities
- C. Computer Programming (*optional*)
 - 1. Determine computer output of given programs
 - 2. Write original programs

VII. Geometry and Measurement

28-31

- A. Geometry
 - 1. Points, lines, and planes
 - a. Points: collinear, noncollinear, coplaner
 - b. Lines, segments, and rays
 - c. Pairs of lines: intersecting, perpendicular, parallel, skew
 - 2. Angles
 - a. Measure of an angle
 - b. Angle pairs: complementary, supplementary, adjacent, vertical
 - 3. Parallel lines cut by a transversal
 - 4. Polygons
 - a. Angle measures of polygons
 - (1) Sum of interior angle measures
 - (2) Regular polygons
 - (3) Exterior angles of polygons (*optional*)
 - b. Triangles and quadrilaterals

5. Geometric solids
 - a. Polyhedrons
 - (1) Faces, edges, and vertices
 - (2) Convex polyhedrons and regular polyhedrons (*optional*)
 - b. Prisms
 - c. Pyramids
 - d. Cones
 - e. Cylinders
 - f. Spheres
- B. Systems of Measurement
 1. Metric system
 - a. Review equivalent measures in metric
 - b. Measures with scientific applications (*optional*)
 - (1) Force: newton (N)
 - (2) Others
 2. Customary system of measurement
- C. Measurement of Geometric Figures
 1. Perimeter
 - a. Polygons
 - b. Circles
 - c. Composite shapes
 2. Surface area
 - a. Rectangular prisms and cubes
 - b. Right circular cylinders
 3. Volume
 - a. Prisms
 - b. Right circular cylinders
 - c. Pyramids
 - d. Cones
 - e. Spheres
- D. Constructions and Activities
 1. Constructions
 - a. Copy a line segment
 - b. Copy an angle
 2. Activities involving constructions and tools of measure
- E. The Pythagorean Theorem
 1. Experiments with triangles
 2. Stating the Pythagorean Theorem
 3. Applications
 4. Constructing irrational lengths (*optional*)
- F. Congruence
 1. Line segments and angles
 2. Congruent figures
- G. Similarity
 1. Similar figures
 2. Applications using proportions
 3. Ratio of perimeters and of areas (*optional*)

SUGGESTIONS TO TEACHERS

I. AN INTRODUCTION TO PROBLEM SOLVING

It is strongly recommended that the mathematics program for grade 8 begin in the same manner as that for grade 7, with a unit devoted to problem-solving techniques and strategies. In this way, the stage is set for the inclusion of problem solving in all other strands of the curriculum.

The students learn problem-solving strategies only through guided experience in solving many problems. Realize that the strategies listed here are not all-inclusive, and that the problems cited are merely samples and suggestions. Teachers should feel free to add other strategies and problems as desired. The best problems, however, come from the students' experience and/or arouse the students' curiosity.

Although the teacher should present various strategies within the context of this unit, the students should be encouraged to find alternative approaches in solving problems. No approach should be cast aside as inappropriate; the discussion of a misguided approach suggested by a student may uncover clues that lead to a correct solution. Remember: STUDENT DETERMINATION OF WHAT TO USE AND WHEN TO USE IT IS AN IMPORTANT ASPECT OF A SOUND PROBLEM-SOLVING ABILITY.

A. Review General Technique

Have the students describe and explain the four basic steps in problem solving:

- (1) Read, or understand, the problem.
- (2) Organize, or formulate, a plan for solving the problem.
- (3) Solve the problem; that is, carry out the plan.
- (4) Check the solution to see if it makes sense.

For a more detailed discussion of problem-solving techniques, see the *Mathematics 7 Syllabus*.

B. Problem-Solving Strategies and Extending a Problem

The problem-solving strategies presented in grade 7 include:

- (1) Use of pictures and diagrams

- (2) Use of a simpler but related problem
- (3) Trial and error (guessing)
- (4) Working backwards
- (5) Patterns

Although the teacher should review these strategies by presenting more advanced problems, the students should be encouraged to find alternative approaches to each solution.

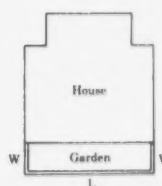
Teachers should consider building a collection of good problems, kept in folders or on index cards, and classified according to strand, concept, or type.

At times, problems lend themselves to the use of more than one strategy, as shown in the first example to follow. In this and in other examples in this section, it is shown that a problem can be extended to higher levels of thinking by altering the question, by seeking a general solution, by noting restrictions to this generalization, and by examining those cases where the general solution does not apply.

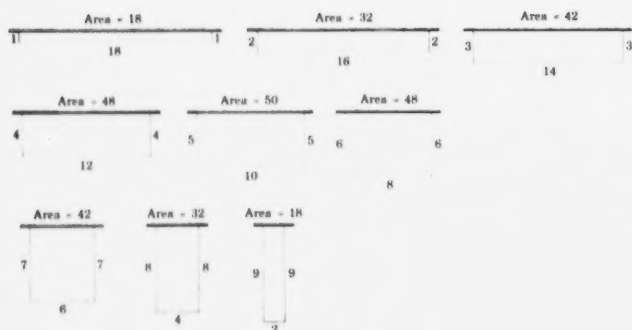
STRATEGY: Use of Pictures and Diagrams (coupled with Patterns)

Example: Thelma plans to build a garden at the side of her house. Since the garden will have a rectangular shape, she needs fencing on only 3 sides of the garden. Thelma has 20 feet of fencing in 1-foot sections. What lengths of fencing should she use to build a garden with the largest possible area?

After the problem is discussed and understood, some students may suggest that pictures be drawn. Note how the use of number patterns guarantees that all possible solutions have been investigated.



The students should realize that $W + W + L = 20$. (The diagrams shown are not drawn to scale. W and L are stated in feet, and area is stated in square feet.)



Answer: $W = 5$ feet $L = 10$ feet or

One length of 10' parallel to the house, and two lengths of 5' each, perpendicular to the house.

Extending The Problem:

Many problems can be extended to higher levels of thinking, identified here as stages 1 through 4.

All students should be able to solve a similar problem using new values, as shown in stage 1. Only some students may be able to form a generalization, as shown in stage 2. However, it is important that the teacher introduce the concept of general solution at this grade level to all.

Studying restrictions to a general solution and cases where the general solution does not apply, shown in stages 3 and 4, provides true challenge and may be treated as optional.

1. Altering the problem or the question:

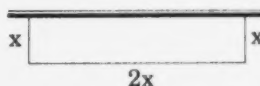
Suppose Thelma had 32' of fencing. Under the same conditions, what lengths of fencing are now needed to build a rectangular garden with the largest possible area?

Answer: $W = 8'$, $W = 8'$, $L = 16'$

2. Seeking a general solution:

Although all students are not expected to prove a generalization at this level, some may find and describe the relationship involved.

Answer: The number of 1-foot sections in L will be double the number of 1-foot sections in W .



3. Noting restrictions to the general solution:

If the lengths of fencing (x , x , and $2x$) are in 1-foot sec-

tions, what restrictions are placed on the total amount of fencing for this general solution to be true?

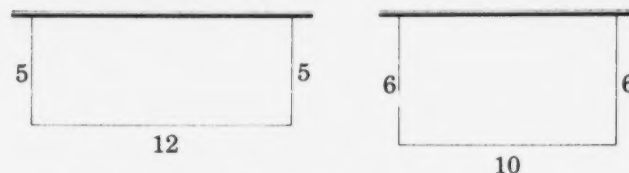
Answer: The total fencing can be 4', 8', 12', 16', etc. That is, the total fencing must be a multiple of 4'.

4. Studying cases where the general solution does not apply:

- a. If fencing is only in 1-foot lengths, study cases where the total available fencing is not a multiple of 4'.

If the total fencing is an even number but not a multiple of 4, there are always two solutions.

For example, total fencing = 22'.



In each case, area = 60 sq. ft.

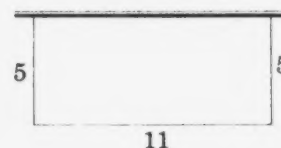
Answer: $W = 5'$, $W = 5'$, $L = 12'$

or

$W = 6'$, $W = 6'$, $L = 10'$

If the total fencing is an odd number, there is only one solution.

For example, total fencing = 21'.

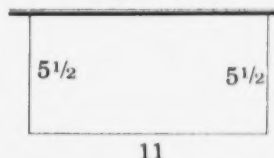


Here, area = 55 sq. ft.

Answer: $W = 5'$, $W = 5'$, $L = 11'$

- b. If fencing, like chicken wire, is *not* restricted to 1-foot lengths, study cases where the total available fencing is not a multiple of 4'. In each case, note that the general solution of x , x , and $2x$ produces a unique answer involving lengths which are not all integers.

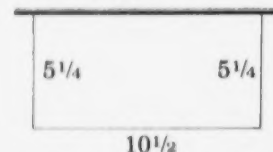
For example,
fencing = 22'.



Here, area = $60\frac{1}{2}$ sq. ft.

Answer: $W = 5\frac{1}{2}'$,
 $W = 5\frac{1}{2}'$,
and $L = 11'$

For example,
fencing = 21'.



Here, area = $55\frac{1}{8}$ sq. ft.

Answer: $W = 5\frac{1}{4}'$,
 $W = 5\frac{1}{4}'$,
 $L = 10\frac{1}{2}'$

STRATEGY: Use of a Simpler but Related Problem (coupled with Patterns)

Example: A teacher seats 28 students in a circle for a discussion. If each student is assigned, in order, a number from 1 through 28, what number is opposite number 8?

It may be difficult to mark 28 equally spaced points on a circle. Can we learn something from a circle with fewer numbers? What circle comes to mind?

A clock face is a circle with the numbers 1 to 12. 12 and 6 are opposite each other. So are 11 and 5, 10 and 4, 9 and 3, etc.. Study these pairs. Some students will discover that each pair of numbers can be subtracted to form a difference of 6. For example, 1 must be opposite 7 on this clock because $7 - 1 = 6$.



Draw another circle in an attempt to find a pattern.

For example, imagine a clock with only eight numbers. Here, 8 is opposite 4, 7 is opposite 3, 6 is opposite 2, etc. The difference of each pair of numbers is 4.



Other simpler clocks could be studied to reinforce the pattern:

With 12 numbers, the opposites have a difference of 6.

With 8 numbers, the opposites have a difference of 4.

With 28 numbers, the opposites have a difference of (?).

The students should recognize that in this last case opposite numbers have a difference of 14. Now, in this numbered circle, what number is opposite number 8?

$$8 - x = 14$$

No positive solution

$$\begin{aligned} x - 8 &= 14 \\ x &= 14 + 8 \\ x &= 22 \end{aligned}$$

(Check: $22 - 8 = 14$)

Answer: 22

Again, note how this problem can be extended.

- 1. Alter the question:** Suppose that there are 30 students in a class. If they are seated in a circle and numbered in order, what number is opposite number 8?

Answer: 23

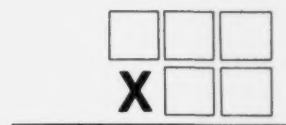
- 2. Seek a general solution:** If the numbers 1 to N are placed in order in a circle, the difference of any two opposite numbers is $\frac{1}{2}N$.

- 3. Note restrictions to the general solution:** N must be an even number. If N is odd, then no two numbers are opposite each other in the circle.

STRATEGY: Trial and Error (Guessing)

Example:

Choose any five single-digit numbers. Form a 3-digit number and a 2-digit number so that, when multiplied, the greatest possible product will result.



Imagine that a student chooses: 7, 3, 5, 8, 1.

(This problem provides computational drill or an excellent opportunity to use a calculator.)

One guess might be: $73 \times 581 = 42,413$

Another guess is: $53 \times 871 = 46,163$ (larger product)

(Some students will make use of estimation, such as $70 \times 800 = 56,000$, or $80 \times 700 = 56,000$. But are we certain of where the 7 and the 8 are to be placed?)

Another guess is: $85 \times 731 = 62,135$ (larger product)

Still another: $73 \times 851 = 62,123$ (oops, smaller product)

And another: $75 \times 831 = 62,325$ (largest so far)

And another: $83 \times 751 = 62,333$ (LARGEST PRODUCT)

Answer: $83 \times 751 = 62,333$

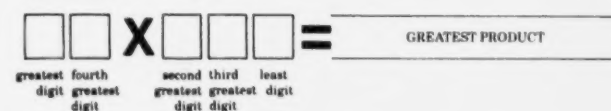
Consider how this problem can be extended.

- 1. Alter the question:** Try five other digits, say 2, 4, 9, 6, 8. Now form a 3-digit number and a 2-digit number so that, when multiplied, the greatest possible product will result.

Answer: $94 \times 862 = 81,028$

- 2. Seek a general solution:** Imagine that the five digits are ordered from the greatest to the least. How should they be placed to get the largest possible product?

Answer:



(This hypothesis can be tested with other digits.)

3. **Other cases:** Will the general rule hold if some of the digits repeat, or must all five digits be different?

Try using 7, 6, 4, 4, 2. The greatest product possible is $74 \times 642 = 47,508$

Answer: The general rule holds if digits repeat.

4. **Alter the question entirely:** Find the *least product*, or find the *greatest quotient* using a 2-digit and a 3-digit number. Try using only four digits to form two 2-digit numbers, or try using six digits.

STRATEGY: Working Backwards

Example: The game of "Force Out" is played with two people. Starting with any whole number, each player in turn must take away a single-digit number. The player who gets 0 is the loser.

How can I develop a rule to win this game?

Try playing the game with two students, or with a student opposing the teacher. Someone in class should recognize that the strategy for winning is developed by thinking backwards:

"If I can subtract and get 1, I can win. If I can subtract and get 11, I can get to 1 because my opponent cannot get to 1. The most he or she can subtract is 9. I can get to 11, if I can get to 21. Again, I cannot let the opposite player get within 9 of 11. This pattern continues to 31, 41,..."

For example, start with:	45
I subtract 4:	41
My opponent subtracts 8:	33
I must get to 31, subtract 2:	31
My opponent subtracts 9:	22
I must get to 21, subtract 1:	21
My opponent subtracts 1:	20
I must get to 11, subtract 9:	11
My opponent subtracts 1:	10
I subtract 9 and win!	1
My opponent must subtract 1:	0

Answer: To win, I must always leave my opponent with a number that has 1 in the units place, such as 1, 11, 21, 31,...

1. **Discuss the game:** If both players know the winning strategy, who will win?

Answer: If the starting number does not have a 1 in the units place, the first player will win. However, if the starting number has a 1 in the units place, the second player will win.

2. **Alter the question:** What winning strategy could be developed if players can only subtract even-numbered digits (2, 4, 6, or 8)?

ADDITIONAL PROBLEMS

There are many other strategies, notably the use of patterns, and general algebraic procedures whereby equations or expressions are written to solve problems. Algebraic techniques, introduced in grade 7, should form an integral part of grade 8. Insight should be encouraged, not rejected, and the students should be allowed to use various strategies to solve problems.

The problems cited above are merely suggestions, as are the examples that follow. The teacher should present a variety of problems to stimulate thinking.

Example: Ann and Andy each had some magazines. If Ann gave 1 of her magazines to Andy, they would each have the same number. However, if Andy gave 1 of his magazines to Ann, then Ann would have twice as many magazines as Andy. How many magazines does each person have?

Answer: Ann has 7; Andy has 5.

Example: Midas has \$1,000,000 and decides to give it away. If he gives away \$50 every hour, how many days would it take to give away the money?

Answer: Exactly $833 \frac{1}{3}$ days or, depending on interpretation, 834 days

Example: A roller coaster takes a group of 24 people every 5 minutes. How long would you have to wait if a roller coaster has just taken off, 72 people are in line in front of you, and every person takes only one ride?

Answer: 20 minutes

Example: Rachel has a 20-foot board that she must cut into pieces of 2', 4', and 5'. If she must have at least one of each size piece, what combination(s) give no waste?

Answer: Three of 2', one of 4', two of 5'; also, one of 2', two of 4', two of 5'

Example: Everybody at a party had soda or cake. If 11 people had soda and 12 people had cake, including 9 who had both, how many people were at the party?

Answer: 14

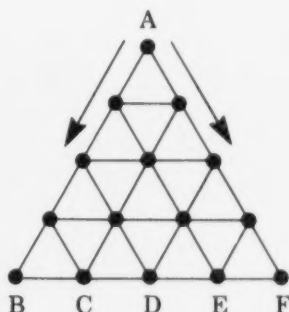
Example: I have 19 coins worth \$1.60 in nickels, dimes, and quarters. How many of each do I have?

Answer: The three possible combinations are as follows...

1 quarter, 9 dimes, 9 nickels;
 2 quarters, 5 dimes, 12 nickels; and
 3 quarters, 1 dime, 15 nickels.

Example:

There are many paths from point A to the points on the line BF. If, at each vertex in the diagram, you may move down to the right or down to the left (but never across horizontally), find the number of paths from A to B, from A to C, from A to D, from A to E, and from A to F.



Answer: In order, there are 1, 4, 6, 4, and 1 paths.

Example: If half of two-thirds of a number is 6, what is the number?

Answer: 18

Example: Find four consecutive integers that have a sum of -2.

Answer: -2, -1, 0, 1

Example: A clock loses five minutes in every hour. If the clock is set at the correct time today, in how many days (or hours) will it next show the correct time?

Answer: Exactly 6 days, or 144 hours

II. NUMBER CONCEPTS

The aim of this unit is to extend the students' proficiency with numbers and their operations. In grade 7, the four fundamental operations with integers were introduced, but work with common fractions and decimal fractions was restricted to positive numbers only. Now, in grade 8, the students should become skillful in handling both positive and negative rational numbers. As the irrational numbers are introduced and defined, it becomes possible to complete the real number line.

Problem-solving situations, primarily with rational numbers, should help the students to see that computational skills are a means to a desirable end rather than an end in themselves. Teachers may integrate all or part of Unit VI, Computer Awareness, within this unit as a basis for computer usage throughout the course, or they may delay the computer unit until later as a way of reinforcing number concepts.

Challenging exercises and the occasional use of hand calculators should help to spark a spirit of inquiry throughout this unit.

A. WHOLE NUMBERS

The students should know that the *natural numbers* are the counting numbers, namely 1, 2, 3, 4, ..., while the *whole numbers* include the natural numbers and 0, that is, 0, 1, 2, 3, 4, ...

1. Sequences and Patterns

It is possible to refresh rather than reteach whole number operations and related terms by using a simple device such as a 9 by 9 multiplication table. Each student should construct such a table in his/her notebook. Patterns and sequences promote this review.

×	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	54
7	7	14	21	28	35	42	49	56	63
8	8	16	24	32	40	48	56	64	72
9	9	18	27	36	45	54	63	72	81

Example: (a) Find the sum of the numbers in row 1. Then the sum for row 2. Then the sum for row 3.

(b) Describe the pattern observed.

- (c) In row 6, find the sum of the numbers in row 6 without actually adding the numbers. How did you do this?

Answer:

- (a) Row 1 $1 + 2 + 3 + \dots + 9 = 45$
 Row 2 $2 + 4 + 6 + \dots + 18 = 90$
 Row 3 $3 + 6 + 9 + \dots + 27 = 135$

- (b) Using addition, each sum is 45 more than the previous sum. Or, using multiplication, $45 = 45 \times 1$, $90 = 45 \times 2$, $135 = 45 \times 3$.

- (c) Using addition, $135 + 45 + 45 + 45 = 270$.

Or, using multiplication, $45 \times 6 = 270$. Show the students the distributive property: $6 + 12 + 18 + \dots + 54 = 6(1 + 2 + 3 + \dots + 9) = 6(45) = 270$.

Or, by doubling every element in row 3 we get row 6. So, $135 + 135 = 270$, or $2 \times 135 = 270$.

Example:

- (a) Select any four numbers in a square block from the table and find their sum. Here, the numbers come from the 3rd and 4th columns, and the 2nd and 3rd rows.

	3	4
2	6	8
3	9	12

- (b) Find (sum of column headings) $\times$ (sum of row headings).
 (c) How are the answers to a and b related, and will this relationship always work?

Answer:

- (a) Sum of numbers $= 6 + 8 + 9 + 12 = 35$.

- (b) Product $=$ (sum of col. head.) $\times$ (sum of row head.)
 $= (3 + 4) \times (2 + 3)$
 $= (7) \times (5) = 35$.

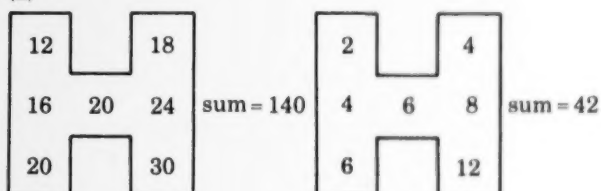
- (c) Sum of numbers $=$ (sum of columns) $\times$ (sum of rows).

Note: Try other examples. The relationship is always true. Extend this concept to rectangular blocks.

Example: (a) Select any seven numbers to form an H-pattern, and find their sum.
 (b) How is the sum of these seven numbers related to the "center" number in the H?

Answer:

(a)



(b) $140 = 7 \times 20$

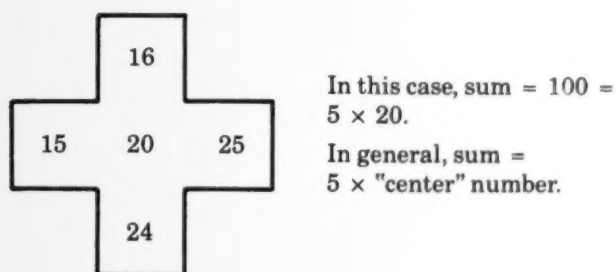
$42 = 7 \times 6$

The sum = 7 times the "center" number.

Note: Investigate an extended H containing eleven numbers.

Example: Find the sum of any five numbers to fit a "cross" pattern in the table. Develop a short-cut to find this sum.

Answer:

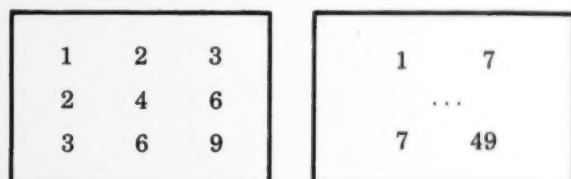


Example: What are the first nine terms in the sequence of square numbers, and where are they found in the table?

Answer: 1, 4, 9, 16, 25, 36, 49, 64, 81

Square numbers lie on the diagonal, upper left to lower right.

Example: Select any square block containing 1 as a corner number. Find relationships involving the corner numbers in the square.



Answer: Many relationships exist; some are shown here.

Proportions:

$$\frac{1}{3} = \frac{3}{9}$$

$$\frac{1}{7} = \frac{7}{49}$$

Cross-products:

$$1 \times 9 = 3 \times 3$$

$$1 \times 49 = 7 \times 7$$

Square roots:

$$\sqrt{9} = 3$$

$$\sqrt{49} = 7$$

Sum is a square:

$$1 + 3 + 3 + 9 = 16$$

$$1 + 7 + 7 + 49 = 64$$

Example: (a) Write the numbers in this sequence: Starting with 1 in the multiplication table, go down two places, then go right one place, go down two places, go right one, etc.

(b) What name do we give to the numbers in this sequence?

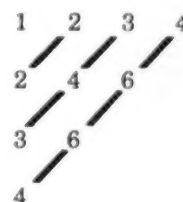
Answer: (a) 1, 3, 6, 10, 15, 21, 28, 36, 45

(b) *Triangular numbers* (See grade 7.)

Example:

(a) Find the sum of the numbers in diagonals 1 2 3 4 (/) in the table. Write the first nine terms in this sequence.

(b) Find the differences between consecutive terms, and identify this sequence of differences.



Answer: (a) 1, 4, 10, 20, 35, 56, 84, 120, 165

(b) 3, 6, 10, 15, 21, 28, 36, 45

The sequence of differences are triangular numbers.

Example: As a challenge for the students, consider the initial diagonal sums. In the sequence 1, 4, 10, 20, 35, ..., it can be shown that

$$1\text{st term} = 1 = \frac{1 \times 2 \times 3}{6}$$

$$2\text{nd term} = 4 = \frac{2 \times 3 \times 4}{6}$$

$$3\text{rd term} = 10 = \frac{3 \times 4 \times 5}{6}$$

(a) Find a general rule to describe the n th term.

(b) Test this rule to find the 5th and 6th terms.

Answer: (a) n th term = $\frac{(n)(n+1)(n+2)}{6}$

$$(b) 5\text{th term} = \frac{(5)(6)(7)}{6} = 35$$

$$6\text{th term} = \frac{(6)(7)(8)}{6} = 56$$

Another sequence introduced in grade 7 was the *Fibonacci numbers*, wherein two consecutive terms are added to produce the next term in the sequence.

Term	1	2	3	4	5	6	7	8	9	10	11	12	13	...
Fibonacci Number	1	1	2	3	5	8	13	21	34	55	89	144	233	...

Example: (a) Circle the first four Fibonacci numbers that have a factor of 2. Then tell which terms these are in the sequence.

(b) Predict the next two terms in the sequence that will have factors of 2, and find these Fibonacci numbers.

Answer: (a) Fibonacci numbers: 2, 8, 34, 144. These represent the 3rd, 6th, 9th, and 12th terms.

(b) Terms 15 and 18 will have factors of 2; these Fibonacci numbers are 610 and 2,584.

Similar questions can be asked regarding other factors in the Fibonacci sequence:

3 is a factor of 3, 21, 144, 987,... (terms 4, 8, 12, 16,...)

5 is a factor of 5, 55, 610, 6765,... (terms 5, 10, 15, 20,...)

Example: Maria collects quarters and half-dollars. In how many different ways can she pick up coins to make each of these amounts: \$.00, \$.25, \$.50, \$.75,..., up to \$2?

Answer: \$.00 – no coins at all 1 way
\$.25 – Q 1 way
\$.50 – QQ, H 2 ways
\$.75 – QQQ, HQ, QH 3 ways
\$1.00 – QQQQ, QQH, QHQ, HQQ, HH 5 ways
... ..

It's the Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34.

The students should be able to find missing terms in a sequence and to explain how they arrived at the answer. Sequences should be altered to match student ability. Only on an *optional* basis should the most capable students be asked to state a rule in terms of n .

Example: Find the missing terms in 5, 7, 9, 11, _____,

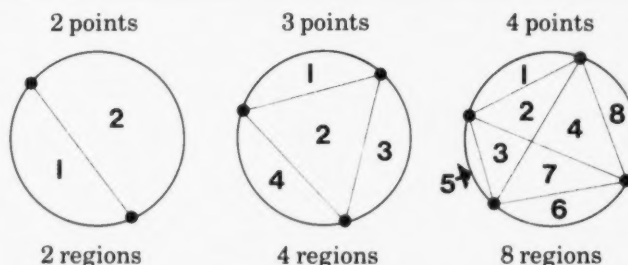
Answer: 13 and 15, found by adding 2 to the previous term. (Note that the n th term is $2n + 3$.)

Example: Find the next term in the sequence.

- (a) 15, 30, 45, 60, _____
- (b) 3, 9, 27, 81, _____
- (c) 3, 3, 6, 9, 15, 24, _____
- (d) 1, 8, 27, 64, _____
- (e) 9, 8, 7, 6, _____
- (f) 4, 9, 25, 49, 121, _____

Answer: (a) 75 (Rule = $15n$)
(b) 243 (Rule = 3^n)
(c) 39 (Fibonacci-type sequence)
(d) 125 (Rule = n^3)
(e) 5 (Rule = $10 - n$)
(f) 169 (Squares of primes)

Example: Find the maximum number of regions formed by joining points on a circle.



If a segment connects two points on a circle, then two regions are formed.

If segments connect three points on a circle, then four regions are formed.

Copy and complete the table:

Number of points	2	3	4	5	6
Number of regions	2	4	8		

Answer: Five points form 16 regions, and six points form 31 regions (*not* 32). This problem illustrates that we can take nothing for granted, and that two or three cases do not lead to a general rule.

2. Powers of Whole Numbers

a. Positive integral and zero exponents

Review the terminology of exponential form. For example, $4^3 = 64$ indicates that:

The base (4) is a factor being repeated.

The exponent (3) tells the number of times the base is a factor.

The power (64) is the product of the repeated factors, $4 \cdot 4 \cdot 4$.

Recall that the first power of a number is equal to the number itself. That is, $N^1 = N$, as in $4^1 = 4$ and $10^1 = 10$. The zero power of a natural number is equal to 1. That is, for any natural number, we define $N^0 = 1$, as in $4^0 = 1$ and $10^0 = 1$.

b. Operations

The study of powers of whole numbers is now extended to include three operations with these powers. Note that powers are restricted here to positive integral and zero exponents (except 0^0), and operations apply to powers of a common base.

(1) *Multiplication*: Begin with a variety of examples in which the students evaluate numerical expressions and form equations.

Example: First, evaluate each expression:

- (a) 3^2 (b) 3^4 (c) $3^2 \cdot 3^4$ (d) 3^6

Then, write an equation using two of the given expressions.

Answer: (a) $3^2 = 3 \cdot 3 = 9$

(b) $3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$

(c) $3^2 \cdot 3^4 = 9 \cdot 81 = 729$

(d) $3^6 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 729$

The equation is $3^2 \cdot 3^4 = 3^6$.

Example: Evaluate each choice to see which equals the product $5^1 \cdot 5^3$

- (a) 5^3 (b) 25^3 (c) 5^4 (d) 10^4

Answer:

- (a) 125 (b) 15,625 (c) 625 (d) 10,000

Since $5^1 \cdot 5^3 = 5 \cdot 125 = 625$, it follows that $5^1 \cdot 5^3 = 5^4$. (Students should write the equation.)

After enough numerical examples, encourage the students to form a generalization based on their observations. For any whole numbers x , a , and b (excluding the expression 0^0), it is true that:

$$x^a \cdot x^b = x^{a+b}$$

The students can now apply this law of exponents to problem-solving situations in which evaluations need not be made.

Example: Find a single term that is equal to $29^8 \times 29^7$.

Answer: 29^{15}

Example: The statement $4^6 \cdot 4^1 = 4^6$ is false. Rewrite the statement to make it true by changing the exponent in only one term.

Answer: $4^6 \cdot 4^1 = 4^7$, or $4^6 \cdot 4^1 = 4^6$, or $4^6 \cdot 4^0 = 4^6$

A similar development (starting with evaluations, leading to a generalization, and concluding with applications and problems) should be followed for the two remaining operations with powers of whole numbers:

(2) *Division*: For any whole numbers x , a , and b (where $a \geq b$ and excluding the expression 0^0), it is true that:

$$x^a \div x^b \text{ or } \frac{x^a}{x^b} = x^{a-b}$$

(3) *Raising to a power*: For any whole numbers x , a , and b (excluding the expression 0^0), it is true that:

$$(x^a)^b = x^{ab}$$

Concepts such as factors, multiples, and divisors may be reviewed while studying this topic.

Example: (a) Evaluate 10^4 .

(b) Write the prime factorization of 10^4 .

Answer: (a) 10,000

(b) $2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5 \cdot 5$ or $2^4 \cdot 5^4$

Example: (a) Which of the four expressions has the smallest value?

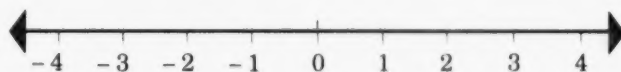
- (1) $2^0 \cdot 2^3$ (2) $(2^3)^2$ (3) $2^3 \cdot 2^2$ (4) $(2^2)^3$

(b) Which of the expressions are divisible by 16?

Answer: (a) (3) (b) All. Note that $16 = 2^4$.

B. INTEGERS

The students should know that the integers consist of the natural numbers, their opposites or additive inverses, and zero. The set of integers can be displayed on a number line.



The positive integers are to the right of 0 on the standard horizontal number line, and above 0 on the vertical number line. Symbols of equality and inequality ($=$, $>$, $<$) are used to compare quantities.

1. Operations

It may be necessary to allow some time for the development of computation skills with integers (see grade 7 for

suggestions). These fundamental operations, however, will play an essential role in all areas of study found in this section on integers, and computation with rational numbers will be extended to include negative values.

a. Absolute value

The *absolute value* of n , written $|n|$, may be expanded to include numerical expressions for n beyond those of a single value. The students may also compute expressions and compare quantities involving more than one absolute value.

Example: Between each pair of numerical expressions, insert one of the symbols ($=$, $>$, $<$) to make the statement true.

- | | |
|-------------------|-----------------|
| (a) $(-5) - 3$ | $ (-5) - 3 $ |
| (b) $(-2) - (-3)$ | $ -2 - -3 $ |
| (c) $ -2 + -5 $ | $ (-2) + (-5) $ |
| (d) $ -7 + 4 $ | $ (-7) + (+4) $ |

Answer:

(a) $(-5) - 3$ $(-5)(3)$ -15	$ (-5) - 3 $ $ +15 $ 15	(b) $(-2) - (-3)$ $-2 + 3$ $+1$	$ -2 - -3 $ $2 - 3$ -1
(c) $ -2 + -5 $ $2 + 5$ 7	$ (-2) + (-5) $ $ -7 $ 7	(d) $ -7 + 4 $ $7 + 4$ 11	$ (-7) + (+4) $ $ -3 $ 3
$-15 < 15$		$+1 > -1$	
$7 = 7$		$11 > 3$	

b. Sequences with Integral Terms

Sequences with integral terms provide another extension of operations with integers, as well as a basis for creative thinking.

Example: (a) Copy and complete this sequence, showing the first eight terms:

1, -2, 4, -8, 16, _____, _____, _____

- (b) Which of these eight terms has the largest value?
- (c) Will the sum of the first 200 terms of this sequence be larger than the answer to part b? Explain why.

Answer: (a) 1, -2, 4, -8, 16, -32, 64, -128

(b) 64

(c) No. Take the terms two at a time and add them. Since each sum is negative, the sum of the first 200 terms must be negative.

Example: (a) Write the next 4 terms in the sequence

5, 8, 3, 6, 1, 4, -1, _____, _____, _____, _____

(b) Give a rule for this sequence.

(c) What is the 50th term in the sequence?

Answer: (a) 2, -3, 0, -5

(b) Use an alternating pattern.

5, 8, 3, 6, 1, 4, -1, 2
+3 -5 +3 -5 +3 -5 +3

or
Use alternating terms.

-2 -2 -2
5, 8, 3, 6, 1, 4, -1, 2
-2 -2 -2

(c) -40

c. Problem Solving

Operations with integers should also be studied within a *problem-solving* mode, with situations ranging from single operations to those involving multiple steps.

Example: The temperature is now 14°C . If the temperature has dropped 1°C per hour for the last 3 hours, what was the reading 2 hours ago?

$$14 - 2(-1) = 14 + 2 = 16$$

Answer: 16°C

Example: An elevator, starting at the 5th floor, went down 2 floors, then down 3, up 7, down 5, and up 1 floor. On what floor is the elevator now stopped?

$$5 - 2 - 3 + 7 - 5 + 1 = 3$$

Answer: the 3rd floor

Example: Mr. Waite lost 5 pounds on the first week of his diet. He lost 3 pounds for each of the next 4 weeks, maintained his weight for 2 weeks, and gained 1 pound during the last week.

(a) Find his average weekly change in weight for this time period.

(b) If Mr. Waite is now at 182 pounds, find his weight at the start of the diet.

Answer: (a) - 2 pounds/week (b) 198 pounds

2. Properties Involving Operations with Integers

The objective of this section should be the recognition of properties, rather than their complete mastery. The students should not be given a list of rules to memorize, but they should be able to formulate these properties after solving exercises and experimenting with integers under the basic operations.

Addition	Subtraction
$[(-3) + (-5)] + (2) \stackrel{?}{=} (-3) + [(-5) + (2)]$ $(-8) + (2) \stackrel{?}{=} (-3) + (-3)$ $-6 = -6$	$(8 - 7) - 3 \stackrel{?}{=} 8 - (7 - 3)$ $1 - 3 \stackrel{?}{=} 8 - 4$ $-2 \neq 4$
After solving other exercises, accept the associative property for addition of integers. $(a + b) + c = a + (b + c)$	The associative property does not hold for subtraction of integers.
Multiplication	Division
$[(-4)(-7)](+3) \stackrel{?}{=} (-4)[(-7)(+3)]$ $(+28)(+3) \stackrel{?}{=} (-4)(-21)$ $84 = 84$	$[(-8) \div 4] \div 2 \stackrel{?}{=} (-8) \div [4 \div 2]$ $(-2) \div 2 \stackrel{?}{=} (-8) \div (2)$ $-1 \neq -4$
After solving other exercises, accept the associative property for multiplication of integers: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$	The associative property does not hold for division of integers.

In a similar fashion, the students should discover that the following properties are true for the set of integers:

a. *Commutative property* for addition and multiplication.

$$a + b = b + a \quad a \cdot b = b \cdot a$$

b. *Associative property* for addition and multiplication.

$$(a + b) + c = a + (b + c) \quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$$

c. Multiplication is *distributive* over addition and subtraction.

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad a \cdot (b - c) = a \cdot b - a \cdot c$$

d. *Identity elements* exist for addition and multiplication.

The identity element for addition is *zero* because, for every integer x :

$$x + 0 = x$$

and

$$0 + x = x$$

The identity element for multiplication is *one* because, for every integer x :

$$x \cdot 1 = x$$

and

$$1 \cdot x = x$$

As an example, consider the general form of the associative property where $*$ represents an operation: $(a*b)*c = a*(b*c)$. After testing many cases, the students should sense that the associative property is true for addition and multiplication of integers. However, a single example using subtraction or division will provide the evidence that the associative property does not hold for these operations. Such an exercise is called a *counterexample* to the rule.

e. An *inverse element* exists for each integer under addition. The additive inverse of x is $(-x)$ because their sum is 0, which is the identity element:

$$x + (-x) = 0 \quad \text{and} \quad (-x) + x = 0$$

f. **Optional:** *Closure* exists under addition, subtraction, and multiplication. If a and b are integers, then their sum $(a + b)$, their difference $(a - b)$, and their product $(a \cdot b)$ must each be an integer.

During this period of investigation, the students should discover that many properties do *not* hold for the set of integers: the commutative and associative properties are not true for either subtraction or division; there are no identity elements for subtraction or division since every integer must remain unchanged when we operate by the identity on the left as well as on the right; inverse elements do not exist under multiplication because for every integer x there is not always an integer y so that $x \cdot y = 1$; the integers do not have closure for division.

The solution of *simple equations*, whether plainly stated or derived from word problems, can highlight the use of some of the properties just studied.

Example: After Mr. Kaye deposited \$50 in his checking account, his balance read \$38. What was Mr. Kaye's checkbook balance before the deposit?

Let x = balance before deposit

$$\begin{array}{rcl} x + 50 = 38 & & x + 50 = 38 \\ x + 50 - 50 = 38 - 50 & \text{or} & -50 = -50 \\ x = -12 & & x = -12 \end{array}$$

(Note that -50 is the additive inverse of $+50$, and that the sum of these numbers is 0, the identity element for addition.)

Answer: $-\$12$

3. Roots

A brief discussion of roots that are integers serves to ease the introduction of roots that are irrational later in this strand. The teacher may, however, delay this topic until the irrational numbers are studied in section D.

a. Two square roots of a positive number

In grade 7, the concept of square root was introduced as an inverse operation of squaring. Since the square of 4 is 16 (written $4^2 = 16$), it follows that the square root of 16 is 4 (written $\sqrt{16} = 4$).

At this time, the students should realize that the square of any nonzero integer is a positive number.

$$\begin{aligned} (+5)^2 &= (+5)(+5) = +25 \\ (-5)^2 &= (-5)(-5) = +25 \end{aligned}$$

By definition, the *square root of a number* is one of two equal factors of the number. An examination of the squares just shown leads to the statement that *every positive number has two square roots*. To indicate that the "square root of 25 is $+5$ or -5 ," a new symbolism is introduced:

$$\pm \sqrt{25} = \pm 5$$

b. Principal square root

The radical sign, $\sqrt{}$, indicates only the *principal square root*, or the positive square root:

$$\sqrt{25} = 5$$

The placement of a negative sign before the radical means that only the opposite of the principal square root is being sought:

$$-\sqrt{25} = -5$$

Provide practice in finding square roots that are integers, primarily with radicands from 1^2 to 25^2 . Vary signs to show the principal square root, two square roots, and the negative square root.

Example:

Evaluate. (a) $\sqrt{49}$ (b) $\pm \sqrt{144}$ (c) $-\sqrt{900}$ (d) $\sqrt[3]{576}$

Answer: (a) 7 (b) ± 12 (c) -30 (d) 24

Example: I am a square root of 36, but I am less than 6. Who am I?

Answer: -6

c. Optional: Principal cube root

The *principal cube root* of a number can be introduced to provide additional practice with signed numbers, but all cube roots should be restricted here to integral values. Note that the principal cube root of a positive number is positive, while the principal cube root of a negative number is negative.

Since $(6)^3 = (6)(6)(6) = 216$, then $\sqrt[3]{216} = 6$.

Since $(-5)^3 = (-5)(-5)(-5) = -125$, then $\sqrt[3]{-125} = -5$.

4. Order of Operations, including Roots

Extend the order of operations studied in grade 7 to now include roots. While a root is primarily regarded as a power or an exponent, it may under certain conditions behave as a grouping symbol, as in $\sqrt{10+6}$. Restate the *order of operations*:

- (1) Work within parentheses first. If more than one symbol of grouping is given, simplify the innermost grouping first.
- (2) Next, simplify all powers (exponents and roots).
- (3) Then, multiply and divide, in order from left to right.
- (4) Lastly, add and subtract, in order from left to right.

Exercises should be kept at a relatively simple level. Note that this topic will reappear in the Geometry strand when problem-solving situations such as those using the Pythagorean Theorem are studied.

Example: Simplify the expression: $\sqrt{64} \div 4 - 5^2$
 $8 \div 4 - 25$
 $2 - 25$

Answer: -23

Example: Between each pair of expressions, insert the symbol ($>$, $<$, $=$) that makes the statement true.

$$\begin{array}{ll} \text{(a)} \sqrt{169-25} & \sqrt{169}-\sqrt{25} \\ \text{(b)} 3\sqrt{49} & \sqrt{49}+\sqrt{49}+\sqrt{49} \end{array}$$

Answer:

$$(a) \frac{\sqrt{169-25}}{\sqrt{144}} = \frac{\sqrt{169}-\sqrt{25}}{13-5}$$

$$12 > 8$$

$$(b) \frac{3\sqrt{49}}{3 \cdot 7} = \frac{\sqrt{49} + \sqrt{49} + \sqrt{49}}{7 + 7 + 7}$$

$$21 = 21$$

C. RATIONAL NUMBERS

The students should know that a *rational number* is any number that can be expressed as the quotient of two integers, a/b , where $b \neq 0$. Each rational number has a position or corresponding point on the number line.

Although operations with rational numbers should be studied through problem-solving situations, it may be necessary to include some time for standard computation.

In grade 7, operations with common fractions and decimal fractions were restricted to the set of positive rational numbers.

1. Operations and Problem Solving

a. Positive and negative common fractions

Basic operations are now extended to the set of negative and positive common fractions. Each operation may be studied using positive rational numbers only as a first step before introducing negative rationals with the operation. A *negative fraction* such as $-6/8$ can be regarded as $-6/+8$ or $+6/-8$, the quotient of a positive integer and a negative integer. Remind the students that *equivalent fractions* are found by reducing or increasing terms, and that these procedures are based on the use of the identity element for multiplication, namely 1 or n/n where $n \neq 0$.

Reducing: $\frac{6}{-8} = \frac{-6}{8} = \frac{-3 \times 2}{4 \times 2} =$

$$\frac{-3}{4} \times \frac{2}{2} = \frac{-3}{4} \times 1 =$$

$$\frac{-3}{4} = \frac{3}{-4}$$

or simply, $\frac{-6}{8} = \frac{-3 \cdot 2}{4 \cdot 2} = \frac{-3}{4}$

Increasing: $\frac{6}{-8} = \frac{-6}{8} = \frac{-6}{8} \times 1 =$

$$\frac{-6}{8} \times \frac{5}{5} =$$

$$\frac{-30}{40} = \frac{30}{-40}$$

Or simply, $\frac{6}{-8} = \frac{6}{-8} \left(\frac{5}{5} \right) = \frac{30}{-40}$

b. Positive and negative decimal fractions

The study of operations with positive and negative common fractions and decimal fractions should include the following:

Estimation of answers;

Common fraction-decimal equivalents;

Combining number forms ($2\frac{1}{4}$ pounds @ \$1.80 per pound);

Rounding.

The use of simple equations related to problem solving may be included where appropriate or may be delayed until the study of properties. Note that operations with rational numbers play an important role in all subsequent sections of this unit.

Example: What is the cost of $\frac{1}{3}$ ounce of insulin if one ounce costs \$29.37? Round the answer to the nearest cent.

(First ask for an estimate: less than \$10.00)

Answer: \$9.79

Example: The sum of three numbers is $7\frac{1}{2}$. If two of these numbers are $4\frac{2}{3}$ and 8, find the third number.

(If an equation is used, let x = the third addend)

$$4\frac{2}{3} + 8 + x = 7\frac{1}{2}$$

$$12\frac{2}{3} + x = 7\frac{1}{2}$$

$$-12\frac{2}{3} + 12\frac{2}{3} + x = -12\frac{2}{3} + 7\frac{1}{2}$$

Answer: $x = -5\frac{1}{6}$

Example: Billy weighs 50 kg and Carl weighs 56 kg. For every kilogram that Billy gains, Carl loses 0.5 kg until they are the same weight. What is that weight?

Answer: 54 kg

Example: In the magic square at the right, the numbers in every row, in every column, and in both diagonals have the same sum.

(a) What is the sum?

(b) Find all missing numbers in the square.

		1
$\frac{1}{4}$	$-\frac{1}{8}$	$1\frac{3}{4}$

Answer: (a) $1\frac{7}{8}$ (b)

$-\frac{1}{2}$	$1\frac{3}{8}$	1
$2\frac{1}{8}$	$\frac{5}{8}$	$-\frac{7}{8}$
$\frac{1}{4}$	$-\frac{1}{8}$	$1\frac{3}{4}$

Example: If it takes $2\frac{3}{4}$ hours to prepare dinner, and it must be served at 6:30 p.m., at what time should the preparation be started?

Answer: 3:45 p.m.

Example: What is the selling price of a \$120 bike if a store plans to sell all bikes at $\frac{1}{3}$ off?

Answer: \$80.

Example: In four days of one week, a local store took in \$90, \$284, \$218, and \$142. On the fifth day, the store paid out \$776 in expenses. What was the average daily gain or loss for this five-day period?

Answer: An average daily loss of \$8.40, or $-\$8.40$

Example: On a number line, B represents -2.6 and C represents $+3$. If B is between A and C, and the distance from B to C is twice the distance from A to B, what number does A represent?



Answer: -5.4

Sequences with rational terms offer another means of extending operations to negative numbers. Verbalizing how the terms of a sequence are found, rather than stating a general rule for the n th term, should be the objective here.

Example: Find the next term in each sequence.

(a) 3, 0.5, -2 , -4.5 , -7 , _____

(b) -4 , 2 , -1 , 2 , -4 , _____

Answer:

(a) -9.5 (Add -2.5 to previous term, or subtract 2.5)

(b) $\frac{1}{8}$ (Multiply previous term by $-\frac{1}{2}$, or divide by -2)

2. Properties Involving Operations with Rational Numbers

The objective here is recognition of properties, not memorization. Discovering rules by testing a series of numerical examples offers the students needed practice in computation.

The properties that held for the set of integers are also true for the set of rational numbers, but a new property exists for this set: *Every rational number except 0 has a multiplicative inverse, also known as a reciprocal.*

The following properties are true for the set of rational numbers:

a. *Commutative property* for addition and multiplication.

$$a + b = b + a$$

$$a \cdot b = b \cdot a$$

b. *Associative property* for addition and multiplication.

$$(a + b) + c = a + (b + c) \quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$$

c. *Distributive property* of multiplication over addition (or subtraction)

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad a \cdot (b - c) = a \cdot b - a \cdot c$$

d. *Identity elements* for addition and multiplication.

Identity for addition is 0; identity for multiplication is 1.

e. *Inverse elements*

Every rational number a/b has an inverse under addition, namely $-a/b$.

Every rational number a/b (except 0) has an inverse under multiplication, namely b/a . The multiplicative inverse is also called the reciprocal.

f. *Optional:* Closure exists under addition, subtraction, and multiplication. It may be observed that the set of nonzero rational numbers is closed under division.

Example: Complete the table using the set of rational numbers.

Number	Additive Inverse	Multiplicative Inverse
$\frac{3}{5}$		
$-2\frac{1}{2}$		
		.04
	0	

Answer:

$\frac{3}{5}$	$-\frac{3}{5}$	$\frac{5}{3}$
$-2\frac{1}{2}$	$+2\frac{1}{2}$	$-\frac{2}{5}$
25	-25	.04
0	0	NONE

Properties of rational numbers can be applied to the solution of *equations* and *inequalities*; each check offers additional practice in computation.

Example: There are 490 girls enrolled in the Stanforth School. If $\frac{2}{5}$ of the students at Stanforth are girls, how many students are enrolled in this school? Let n = number of students enrolled at Stanforth.

Then $\frac{2}{5}n$ = number of girls.

$$\frac{2}{5}n = 490$$

$$\frac{5 \cdot 2}{2 \cdot 5}n = \frac{5}{2} \cdot 490$$

$$n = 1,225$$

Check:

$$\frac{2}{5}(1225) = 490$$

$$490 = 490$$

(True)

Answer: 1,225 students

3. Powers and Roots of Rational Numbers

a. Powers with positive integral and zero exponents

The evaluation of *powers* of rational numbers, when restricted to *positive integral and zero exponents*, offers another form of computation.

$$\left(\frac{2}{7}\right)^2 = \frac{2}{7} \times \frac{2}{7} = \frac{4}{49}$$

$$\left(-\frac{1}{2}\right)^3 = \left(-\frac{1}{2}\right) \left(-\frac{1}{2}\right) \left(-\frac{1}{2}\right) = -\frac{1}{8}$$

$$\left(1\frac{1}{2}\right)^3 = \left(\frac{3}{2}\right)^3 = \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} = \frac{27}{8} = 3\frac{3}{8}$$

The students should realize that for any rational number a/b and for any positive integral exponent y :

$$\left(\frac{a}{b}\right)^y = \frac{a^y}{b^y}$$

For any rational number except 0, we define $\left(\frac{a}{b}\right)^0 = 1$.

Example: For which of the following numbers will its square be less than the number itself?

$$(1) -\frac{1}{5} \quad (2) -0.4 \quad (3) \frac{1}{3} \quad (4) 3.98$$

Answer: (3) $\frac{1}{3}$

b. Square roots that are rational numbers

The evaluation of *roots* of rational numbers should be limited to square roots (or principal cube roots if discussed earlier) which, when computed, are rational numbers.

$$\sqrt{\frac{25}{49}} = \frac{5}{7} \quad \pm \sqrt{.04} = \pm .2$$

$$\sqrt[3]{\frac{-27}{125}} = -\frac{3}{5}$$

A radicand, given in decimal form, may be converted to a common fraction before its root is evaluated. For example,

$$\sqrt{.04} = \sqrt{\frac{4}{100}} = \frac{2}{10} = .2, \text{ or}$$

$$\sqrt{.04} = \sqrt{\frac{4}{100}} = \sqrt{\frac{1}{25}} = \frac{1}{5} = .2$$

Example: Which two roots are equal in value?

$$\sqrt{\frac{4}{9}}, \sqrt{\frac{9}{4}}, \sqrt{\frac{16}{25}}, \sqrt{.49}, \sqrt{2.25}$$

Answer: $\sqrt{\frac{9}{4}}, \sqrt{2.25}$ (Each equals $\frac{3}{2}$ or 1.5)

4. Negative Integral Exponents, with Natural Number Bases

Negative exponents may be introduced by examining powers in descending order. As the students evaluate the powers, an intuitive feeling for patterns and successive divisions usually develops.

$10^3 = 1000$	$2^4 = 8$
$10^2 = 100$	$2^3 = 4$
$10^1 = 10$	$2^2 = 2$
$10^0 = 1$	$2^1 = 1$
$10^{-1} = \frac{1}{10}$	$2^{-1} = \frac{1}{2}$
$10^{-2} = \frac{1}{100}$	$2^{-2} = \frac{1}{4}$
$10^{-3} = \frac{1}{1000}$	$2^{-3} = \frac{1}{8}$

Negative exponents may also be presented by extending the rules for multiplication and division of powers to include negative integral exponents.

Example: Since $\frac{100}{1000} = \frac{10 \times 10}{10 \times 10 \times 10} = \frac{10^2}{10^3} = 10^{-1} =$

$$10^{-1} \text{ and } \frac{100}{1000} = \frac{1}{10}, \text{ thus } 10^{-1} = \frac{1}{10}$$

Example: Since $\frac{4}{32} = \frac{2 \times 2}{2 \times 2 \times 2 \times 2 \times 2} = \frac{2^2}{2^5} =$

$$2^{2-5} = 2^{-3} \text{ and } \frac{4}{32} = \frac{1}{8}, \text{ thus } 2^{-3} = \frac{1}{8}$$

(Observe that $\frac{1}{8} = \frac{1}{2^3}$. Therefore, $2^{-3} = \frac{1}{2^3}$.)

In more general terms, for every natural number x and for every negative integral exponent $(-a)$, it is true that:

$$x^{-a} = \frac{1}{x^a}$$

A limited number of exercises may be provided for computation using natural number bases other than 10:

Example:

Evaluate (a) 8^{-2} (b) $\frac{7^{10}}{7^{11}}$ (c) $12^2 \times 12^{-1}$

Answer: (a) $\frac{1}{64}$ (b) $\frac{1}{7}$ (c) 12

It is essential, however, that the students have a sense of negative integral powers of 10 to reinforce place value in our number system and to lead to scientific notation.

10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}
thousands	hundreds	tens	ones	tenths	hundredths	thousandths	ten-thousandths

Example: Write 10^{-3}

(a) as a fraction (b) as a decimal

Answer: (a) $\frac{1}{1000}$ (b) .001

5. Scientific Notation

Very large and very small positive numbers are often stated in scientific notation, a form that is not unfamiliar to the students who use computers or hand calculators.

In *scientific notation*, a number is given as the product of two quantities: the first is a number equal to or greater than 1, but less than 10; the second is an integral power of 10. The students should be able to convert from decimal notation to scientific notation, and vice-versa.

a. Numbers larger than or equal to 1

Review scientific notation for *numbers larger than or equal to 1*, citing examples from reference books where possible:

$$27,000,000 = 2.7 \times 10,000,000 = 2.7 \times 10^7$$

$$4,921,000 = 4.921 \times 10^6 \quad 3 = 3 \times 10^0$$

Example: The radius of the observable universe is ap-

proximately 13,000,000,000 light-years. State this number in scientific notation.

Answer: 1.3×10^{10}

Example: Write "one hundred billion" in (a) decimal notation (b) scientific notation

Answer: (a) 100,000,000,000 (b) 1×10^{11}

b. Numbers less than 1 but greater than 0.

Introduce scientific notation for positive but small decimal fractions, that is, *numbers less than 1 but greater than 0*. Again, cite scientific references for examples.

$$.7 = 7 \times .1 = 7 \times 10^{-1} \quad .02 = 2 \times .01 = 2 \times 10^{-2}$$

$$.0000041 = 4.1 \times .000001 = 4.1 \times 10^{-6}$$

Example: The smallest known beetle on earth is less than 0.0085 inches in length and weighs about 0.000005 grams. Express each decimal number in scientific notation.

Answer: $0.0085 = 8.5 \times 10^{-3}$

$$0.000005 = 5 \times 10^{-6}$$

c. Optional: Operations involving numbers in scientific notation

Operations involving numbers in scientific notation are important in our technical world. Examples of *multiplication* and *division* may be presented, and the level of difficulty varied to match the ability and interest of the class. In initial examples, show both decimal notation and scientific notation.

	Decimal Notation	Scientific Notation
Example:	$4 \times 21,000$ 84,000	$4 \times (2.1 \times 10^4)$ $(4 \times 2.1) \times 10^4$ 8.4 $\times 10^4$
Example:	$720,000,000 \div 3$ 240,000,000	$(7.2 \times 10^8) \div 3$ 2.4×10^8

Example: One light-year equals approximately 6×10^{12} miles. A star is four light-years from Earth. In scientific notation, how many miles away from Earth is this star?

$$4 \times (6 \times 10^{12}) = (4 \times 6) \times 10^{12} = 24 \times 10^{12} = (2.4 \times 10^1) \times 10^{12} = 2.4 \times 10^{13}$$

Answer: 2.4×10^{13}

6. Rational Numbers Defined

To conclude this unit and to prepare for the next topic, review certain definitions:

A rational number is any number that can be expressed as the quotient of two integers, a/b , where $b \neq 0$.

In effect, every rational number can be expressed as a common fraction. Therefore: *Every rational number can be written as either a terminating decimal or an infinitely repeating decimal.*

$$\frac{1}{4} = .25 \quad \frac{-17}{5} = -3.4 \quad \frac{1}{6} = .16666... = .1\bar{6}$$

Optional: The process to convert repeating decimals to common fractions may be demonstrated algebraically.

Example: Convert $0.363636...$ or $0.\overline{36}$ to a fraction in lowest terms.

$$\begin{array}{rcl} \text{Let } N = 0.3636... = 0.\overline{36} \\ \text{Then } 100N = 36.3636... = 36.\overline{36} \\ 100N = 36.3636... \quad \text{or} \quad 100N = 36.\overline{36} \\ - N = 0.3636... \\ \hline 99N = 36 \end{array} \quad \begin{array}{rcl} - N = .\overline{36} \\ \hline 99N = 36 \end{array}$$

Answer: $\frac{4}{11}$

$$N = \frac{36}{99} = \frac{4}{11}$$

$$N = \frac{36}{99} = \frac{4}{11}$$

D. IRRATIONAL NUMBERS

1. Definition and Examples

An irrational number is a number that, when expressed as a decimal, does not terminate and does not repeat.

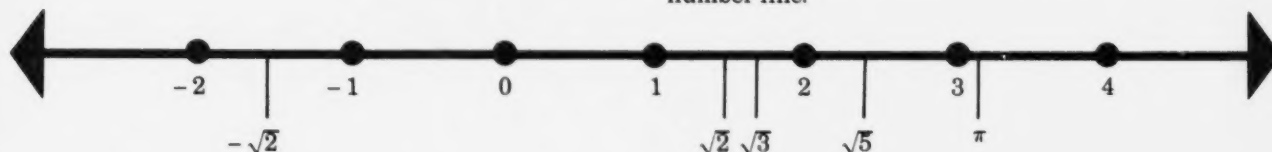
If a decimal does not repeat (or terminate), then the

number cannot be written as a common fraction. One example of an irrational number is

$0.123456789...$

Other irrational numbers include $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $-\sqrt{2}$, and π .

Every irrational number corresponds to a point on the number line.



Lengths such as $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$ can be constructed by using the Pythagorean Theorem, (see Unit VII – Geometry and Measurement). The teacher may integrate geometric concepts at this point or concentrate on finding *rational approximations* for these irrational numbers.

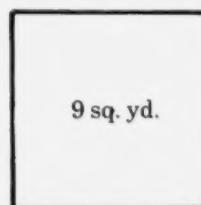
The values often associated with π , such as $22/7$ or 3.14 or 3.1416 , are again simply rational approximations of this irrational number.

Most applications involving irrational numbers have a basis in geometric situations.

2. Rational Approximation and Square Root

There are many ways that can be used to find a rational approximation of a square root, including tables, the square root algorithm, and the calculator. However, it is recommended that one or both of the following methods to approximate square root be demonstrated in class to provide an explanation of the underlying concept. A hand calculator *without* a square root key might prove to be a useful tool. Cite a physical model as motivation.

A square carpet has an area of 9 square yards.



9 sq. yd.

$$s^2 = A$$

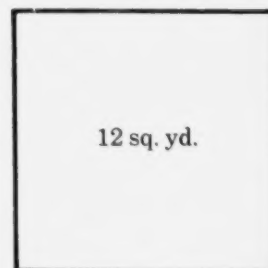
$$s^2 = 9$$

$$\sqrt{s^2} = \sqrt{9}$$

$$s = 3$$

Each side of the carpet measures 3 yards. The number is a rational number.

A square carpet has an area of 12 square yards.



12 sq. yd.

$$s^2 = A$$

$$s^2 = 12$$

$$\sqrt{s^2} = \sqrt{12}$$

$$s = \sqrt{12}$$

Each side has a length in yards of $\sqrt{12}$. The number $\sqrt{12}$ is an irrational number. What is an *approximate* value for $\sqrt{12}$ to the nearest tenth?

a. Trial and error

Consider a table of squares. Since 12 is between 9 and 16, or $9 < 12 < 16$, then $\sqrt{9} < \sqrt{12} < \sqrt{16}$ or $3 < \sqrt{12} < 4$.

- (1) To obtain a closer approximation, guess values using tenths, and square each guess.
- | | | |
|--------------|--------------|--------------|
| 3.3 | 3.4 | 3.5 |
| $\times 3.3$ | $\times 3.4$ | $\times 3.5$ |
| 10.89 | 11.56 | 12.25 |

$$3.4 < \sqrt{12} < 3.5$$

- (2) Since 12.25 is closer to 12 than is 11.56, then 3.5 is closer to $\sqrt{12}$ than is 3.4.

Answer: $\sqrt{12} = 3.5$ (nearest tenth)

NOTE: The process can be continued by guessing values using hundredths

$$3.46 < \sqrt{12} < 3.47$$

3.46	3.47
$\times 3.46$	$\times 3.47$
11.9716	12.0409

$$\sqrt{12} = 3.46 \text{ (nearest hundredth)}$$

b. Divide and average

Compare a known square root to some division problems.

$$\begin{array}{r} \sqrt{36} = 6 \\ 6 \\ 6 \overline{)36} \end{array}$$

In division, when the quotient is equal to the divisor, each of these numbers is a square root. Since $36 \div 6 = 6$, it follows that $\sqrt{36} = 6$.

$$\begin{array}{r} 9 \\ 4 \overline{)36} \end{array}$$

However, if the divisor is less than the square root, then the quotient will be greater than the square root, and vice-versa. Since the square root lies between the di-

visor and the quotient, a better approximation of the square root will be the average of these two numbers.

Example: Find $\sqrt{12}$ to the nearest tenth.

- (a) Guess 3, and divide

$$\begin{array}{r} 4 \\ 3 \overline{)12} \end{array} \quad \text{3 is too small; 4 is too large}$$

- (b) Find the average

$$\frac{3 + 4}{2} = \frac{7}{2} = 3.5$$

- (c) Divide again, using the average

$$\begin{array}{r} 3.42 \\ 3.5 \overline{)12.000} \end{array}$$

- (d) Find the average

$$\frac{3.5 + 3.42}{2} = \frac{6.92}{2} = 3.46$$

- (e) Round to the nearest tenth

$$3.46 = 3.5 \text{ (nearest tenth)}$$

Answer: $\sqrt{12} = 3.5$ (nearest tenth)

Example: Two sticks, each exactly $\sqrt{3}$ meters in length, were placed end to end. Hugo reasoned that $\sqrt{3} + \sqrt{3} = \sqrt{6}$, or that the total length is $\sqrt{6}$. Was he correct? Explain why.

Answer: $\sqrt{3} + \sqrt{3} \approx 1.7 + 1.7 = 3.4$ ($\approx$ means approximately equals)

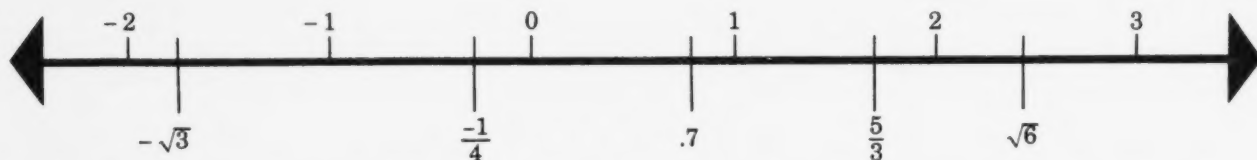
$$\text{But } \sqrt{6} \approx 2.4$$

$$\text{No, } \sqrt{3} + \sqrt{3} \neq \sqrt{6}.$$

E. REAL NUMBERS

1. Definition and Subsets of the Real Numbers

The set of numbers consisting of all rationals and irrationals is called the set of *real numbers*. In other words, the real numbers consist of decimals that repeat or terminate, and decimals that do not repeat.

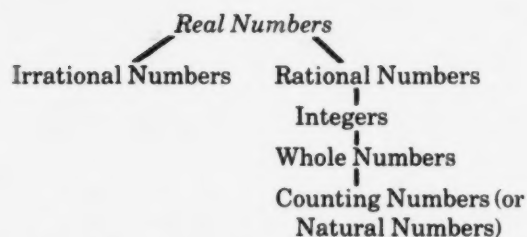


Every real number is represented by a point on the number line, and every point represents a real number. There is a *one-to-one correspondence* between the points on the number line and the set of real numbers. For this reason, the number line is called the *real number line*.

The properties identified for the rational numbers are true for the set of real numbers, but they will not be tested at this level.

The students have seen sets of numbers through a process of extension: the counting numbers or natural numbers $\rightarrow$ the whole numbers $\rightarrow$ the integers $\rightarrow$ the rational numbers $\rightarrow$ the real numbers.

By studying this flow in a reverse pattern, the *subsets of the real numbers* are presented and displayed.



The union of the irrational numbers and the rationals is the set of real numbers, but the intersection of these sets is empty:

$$\text{Irrationals} \cup \text{Rationals} = \text{Real Numbers}$$

$$\text{Irrationals} \cap \text{Rationals} = \{ \quad \}$$

The natural numbers form a subset of the whole numbers, a subset of the integers, a subset of the rationals, and a subset of the reals. Other sets follow similar patterns:

Integers $\subseteq$ Rationals "The integers are a subset of the rationals."

Integers $\subset$ Rationals "The integers are a proper subset of the rationals."

A general understanding of the subsets of the real numbers should result from class discussion.

2. The Real Number Line

Now that *the number line is complete*, some mention of number line graphs should be made. Problem-solving situations using algebraic techniques may be discussed at this point, or reserved until the Algebra strand is presented. The students should have some sense of the domain of numbers applicable to each problem.

Example: A curtain rod is less than 24 inches in length. The length of two of these rods is at least 40 inches. Find

all possible lengths for the curtain rod.

Let x = length of one curtain rod in inches

$$2x \geq 40$$

$$x \geq 20 \quad \text{and} \quad x < 24$$

Answer: Any real number greater than or equal to 20, but less than 24.



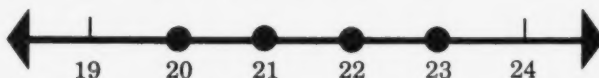
Example: Paul had fewer than 24 fish in a large tank. The fish population doubled because babies were born. If there are now 40 or more fish in the tank, how many fish were there before the babies were born?

Let x = number of fish before babies

$$2x \geq 40$$

$$x \geq 20 \quad \text{and} \quad x < 24$$

Answer: 20, 21, 22, or 23



Example: An apple that weighs 80 grams is cut into two parts that are not equal in weight. Show the weight of the smaller part on the number line.

Answer: $0 < \text{weight} < 40$. The part is less than 40g, but more than 0.



III. RATIO, PROPORTION, AND PERCENT

This strand offers the students many opportunities to practice problem-solving strategies while improving computational skills. The unit also serves as a basis for the integration of topics in the *Syllabus*.

Definitions, forms of notation, and basic concepts presented in grade 7 should be reviewed before more advanced applications are studied. Extensions in grade 8 include percents greater than 100% and less than 1%; percent of increase and decrease; compound interest; trigonometry of the right triangle; and circle graphs.

A. RATIO AND PROPORTION

1. Ratio and Rate

A *ratio* is a comparison of two numbers by division. In the comparing of numbers denoting measurements, the terms of the ratio must be expressed in the same units of measure to show the ratio as a number.

Example: Express, in simplest form, the ratio of 15 minutes to $1\frac{1}{2}$ hours.

Method 1:

$$\frac{15 \text{ min.}}{1\frac{1}{2} \text{ hrs.}} = \frac{15 \text{ min.}}{90 \text{ min.}} = \frac{15}{90} = \frac{1}{6}$$

Method 2: (Note the simplification of a complex fraction.)

$$\frac{15 \text{ min.}}{1\frac{1}{2} \text{ hr.}} = \frac{\frac{1}{4} \text{ hr.}}{1\frac{1}{2} \text{ hr.}} = \frac{\frac{1}{4}}{1\frac{1}{2}} = \frac{\frac{1}{4} \times 4}{\frac{3}{2} \times 4} = \frac{1}{6}$$

Answer: The ratio, read as "1 to 6," may be written as $\frac{1}{6}$ 1 to 6 1:6

A *rate* is a ratio in which the numbers denote measurements that are not expressed in the same units. Examples of rates include "80 km/hr." and "3 sale items per customer."

In some cases, one can express the numbers of a rate in the same units of measure, but this is not done because the resulting ratio would be meaningless. For an example, an employee may be granted 10 paid sick days each year, and there are 255 workdays in a year. Here, the rate

of 10 sick days/year makes sense. However, the ratio of 10/255 or 2/51 has no meaning per se.

Ratios and rates can be used to make comparisons.

Example: A country has a population of 10,000,000 people and covers an area of 125,000 sq. miles. Another country has a population of 2,500,000 people and covers an area of 30,000 square miles. Which country seems to be more crowded?

$$\frac{10,000,000}{125,000} = \frac{80}{1} \text{ or 80 people per square mile, population density 80:1}$$

$$\frac{2,500,000}{30,000} = \frac{83.3}{1} \text{ or 83.3 people per square mile, population density 83.3:1}$$

Answer: The second country is more crowded. (Note the use of the base number 1 in stating these rates. Explain that approximate or rounded numbers are used to present the comparison in an easily understood form.)

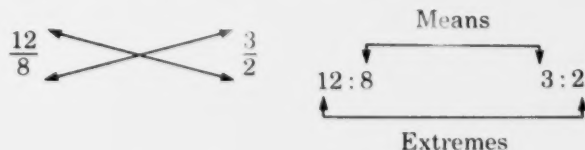
2. Proportions and Comparisons

a. Cross-products rule and equations

A *proportion* is a statement showing that two ratios are equal. The following proportion, read as "12 is to 8 as 3 is to 2," is shown by writing either of two common forms:

$$\frac{12}{8} = \frac{3}{2} \qquad 12:8 = 3:2$$

In either form, the first and fourth terms (here, 12 and 2) are called the *extremes*, and the second and third terms (here, 8 and 3) are called the *means*. In a proportion, the product of the means is equal to the product of the extremes. When the ratios are written as fractions, this relation is sometimes called the cross-products rule.



$$8 \times 3 = 12 \times 2, \text{ or } 24 = 24$$

The cross-products rule should be stated in general terms, noting that division by 0 is excluded in each ratio:

$$\frac{a}{b} = \frac{c}{d} \quad \text{if and only if } ad = bc \quad (b \neq 0, d \neq 0)$$

$$a:b = c:d \quad \text{if and only if } ad = bc \quad (b \neq 0, d \neq 0)$$

Optional: An algebraic proof of the statement may be presented. Since $b \neq 0$ and $d \neq 0$, the product $bd \neq 0$.

$$\begin{array}{l|l} \text{If } \frac{a}{b} = \frac{c}{d} & \text{If } ad = bc \\ \frac{a}{b}(bd) = \frac{c}{d}(bd) & \frac{ad}{bd} = \frac{bc}{bd} \\ \text{then } ad = bc & \text{then } \frac{a}{b} = \frac{c}{d} \end{array}$$

Equations are used to solve proportions with one unknown term. The known ratio in the proportion may be reduced to lower terms or increased to higher terms before writing the equation formed by the cross-products. (Many other applications follow in later sections of this unit.)

Example: If 8 pencils cost 48 cents, find the cost of one dozen pencils.

$$\begin{array}{lcl} \frac{48}{8} = \frac{x}{12} & \text{or} & \frac{6}{1} = \frac{x}{12} \\ 8x = 576 & & 1x = 72 \\ x = 72 & & \end{array}$$

Answer: 72 cents

b. Order relations and inequalities

If the product of the means is not equal to the product of the extremes, then the proportion is not true. For example:

$$\frac{2}{3} \neq \frac{4}{5} \text{ because } 2 \times 5 \neq 3 \times 4, \text{ or } 10 \neq 12$$

Whenever an equality does not exist, symbols of inequality indicate the greater ratio. Numerical comparisons can be made by using equivalent fractions or, on an optional basis, the order relations:

$$\frac{a}{b} > \frac{c}{d} \text{ if and only if } ad > bc \quad (b \neq 0, d \neq 0)$$

$$\frac{a}{b} < \frac{c}{d} \text{ if and only if } ad < bc \quad (b \neq 0, d \neq 0)$$

Example: Which symbol ($>$, $<$, $=$) placed between $\frac{2}{3}$ and $\frac{4}{5}$ shows a true relation?

Method 1:
Equivalent fractions

$$\frac{2}{3} ? \frac{4}{5}$$

$$\frac{2}{3} \times \frac{5}{5} ? \frac{4}{5} \times \frac{3}{3}$$

$$\frac{10}{15} < \frac{12}{15}$$

$$\text{Answer: } \frac{2}{3} < \frac{4}{5}$$

Method 2:
Order relations (Optional)

$$\frac{2}{3} ? \frac{4}{5}$$

$$2 \times 5 ? 3 \times 4$$

$$10 < 12$$

$$\text{Answer: } \frac{2}{3} < \frac{4}{5}$$

Realistic problems often demand genuine computational ability, and judgment as well. Alternative approaches to such problems should be presented and discussed.

Example: A $5\frac{1}{2}$ -ounce bar of Kleen soap costs \$.39, and a 7-ounce bar of the same soap costs \$.53. Which size is the better buy?

Method 1: Standard division

$$\frac{.39}{5\frac{1}{2}} ? \frac{.53}{7}$$

If taken to two places, the comparative worth of fractions must be known:

$$.07\frac{1}{11} < .07\frac{4}{7}$$

Or, extend each quotient to three places:

$$.070 + < .075 +$$

Method 2: Hand calculator

In addition to entering the data correctly, a student should record each quotient and form a comparison on the basis of the first digits that differ. Here, 8 place decimals are recorded, with differences noted in the thousandths place:

$$.07090909 < .07571429$$

Method 3: Order relations

$$\frac{.39}{5\frac{1}{2}} ? \frac{.53}{7}$$

$$.39 \times 7 ? .53 \times 5\frac{1}{2}$$

$$2.73 < 2.915$$

$$\text{Therefore, } \frac{.39}{5\frac{1}{2}} < \frac{.53}{7}$$

Answer: The $5\frac{1}{2}$ -ounce bar is the better buy.

3. Problem Solving with Proportions

Although applications involving proportions need not be limited to those cited here, the following topics should be covered as a minimum. Individual or small group projects may be assigned to give the students experience in practical applications.

a. Consumer problems

Consumer problems, including comparison shopping and unit pricing.

Project: From a newspaper or a cookbook, copy a recipe that serves 4 people. Rewrite the recipe, correcting all measurements, so that it will serve 6 people. (A more challenging exercise is to cut a recipe down to serve fewer people, since the measurements must be compatible with standard kitchen measuring devices.)

Project: From a paint label or from a clerk in a store, find

the surface area that can be covered by 1 gallon of paint. Find the area of a section of the school such as the cafeteria and, using proportions, determine the number of gallons needed to paint that section.

Project: From a drugstore, find out the amount of medication in a bottle of insulin and the size of the average dosage that a person with diabetes would need to take. How many doses are in the bottle?

b. Rates

Rates, including those related to distance and time. The students should understand the formula $D = RT$ (distance = rate $\times$ time) and the related formulas, $R = D/T$ and $T = D/R$. Other rates, such as population density, should be discussed.

Example: A car travels at 45 m.p.h. How far will the car travel in $6\frac{1}{2}$ hours at this rate?

$$\frac{45 \text{ miles}}{1 \text{ hour}} = \frac{n \text{ miles}}{6\frac{1}{2} \text{ hours}} \quad \text{or} \quad 45:1 = n:6\frac{1}{2}$$

Answer: 292.5 miles

Project: Find the pulse rate for several people. Learn how to take a pulse, count the number of beats in 15 seconds, and use a proportion to state each pulse rate as the number of beats per minute.

c. Scale drawing

Scale drawing, including learning how to read maps and diagrams that are based on scale drawings.

Example: An architect drew a floor plan of a school, using a length of 1 centimeter to correspond to an actual length of 2 meters.

- (1) Find the ratio of any length in the drawing to its actual size.
- (2) Assume that the actual length of a room is 8.4 m. Find the corresponding length in centimeters in the drawing.
- (3) Assume that the width of a room is 5.3 cm in the floor plan. Find its actual width in meters.

Analysis

- | | | |
|--------------|---|---|
| (1) 1 cm: 2m | (2) Change all lengths to centimeters: | (3) Change all lengths to meters: |
| 1 cm: 200 cm | | |
| 1: 200 | $\frac{1}{200} = \frac{L \text{ (cm)}}{840 \text{ (cm)}}$ | $\frac{1}{200} = \frac{.053 \text{ (m)}}{W \text{ (m)}}$ |
| | Or use: | Or use: |
| | $\frac{1 \text{ cm}}{2 \text{ m}} = \frac{L \text{ cm}}{8.4 \text{ m}}$ | $\frac{1 \text{ cm}}{2 \text{ m}} = \frac{5.3 \text{ cm}}{W \text{ m}}$ |

Answer:

- (1) 1:200 or $\frac{1}{200}$ (2) 4.2 cm (3) 10.6 m

Optional: Constructing maps and diagrams to scale

Project: Build a model from a kit. Make a chart showing two columns of figures: dimensions of the model, and corresponding dimensions of the actual object.

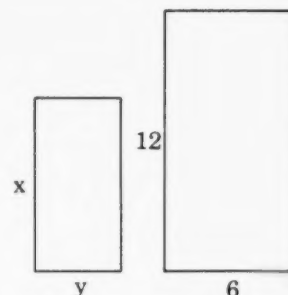
Project: Cut a map into a number of rectangular pieces (12, 18, 30, etc.) and distribute one piece to each member of the group. Using a ratio of 1:10, each student should enlarge his/her section so that each length in the new drawing is 10 times its length in that of the original map. Tape the newly drawn pieces together, and compare the enlargement with the original.

d. Similar polygons, especially triangles

Although similarity (as well as congruence) is treated with greater detail in the Geometry strand of grade 8, the students should see at this point some simple geometric applications of proportions. Emphasis should be placed on similar triangles, recognition of corresponding parts, and finding the missing length of a side by using a proportion. Integrate at this stage any or all of the appropriate content from the Geometry strand.

Two polygons are said to be *similar* when the lengths of the sides of one polygon are proportional to the lengths of the corresponding sides of the second polygon. In simpler terms, one polygon is like a scale drawing of the other.

Example: Two rectangles are similar, and their corresponding sides are in the ratio of 3:2. The larger rectangle is 12 by 6. Find the length and width of the smaller rectangle.

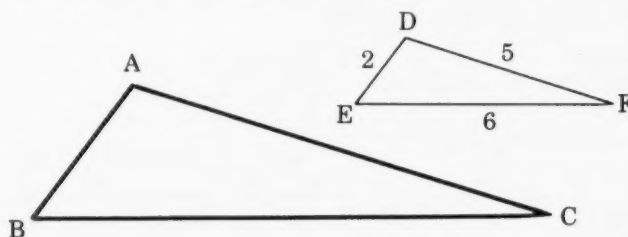


$$\frac{12}{x} = \frac{3}{2} \quad \frac{6}{y} = \frac{3}{2}$$

Answer: The dimensions are 8 by 4.

Example: Triangle ABC is similar to triangle DEF, and $AB:DE = 3:1$.

- (1) Write three proportions, each of which compares the length of corresponding sides and uses the ratio 3:1.
- (2) Find the missing lengths in triangle ABC.



Answer: (1) $AB:DE = 3:1$, or $AB:2 = 3:1$
 $BC:EF = 3:1$, or $BC:6 = 3:1$
 $CA:FD = 3:1$, or $CA:5 = 3:1$
 (2) $AB = 6$, $BC = 18$, $CA = 15$

Example: Juan is 1.6 meters tall. At the same time that Juan's shadow is 0.5 meter, the shadow of a tree is 2 meters. How tall is the tree?

$$\frac{1.6}{0.5} = \frac{x}{2}$$



Answer: 6.4 m

Project: Apply the concepts from the previous example to indirectly measure the heights of objects such as the school building or a flagpole.

B. Right Triangle Trigonometry

A natural outgrowth of the study of similar triangles and ratios is trigonometry. This unit may be presented here or at the conclusion of the unit on Geometry. Trigonometry offers many opportunities for integrating arithmetic, algebra, and geometry.

1. Indirect Measurement

When a ruler is used to measure the length of a line segment, or a protractor is used to measure an angle, a *direct measure* of the segment or the angle is taken.

In some cases, it is impossible or inconvenient to measure an object directly: How wide is a river? How high is a plane above the ground? Here, an *indirect measurement* is made, based on a formula and some computation.

The students should realize that indirect measurement is not exact. Since every indirect measure is based on some direct measures used in a formula, the indirect measure can be no more precise than the direct measure used. When finding indirect measures, rounding is a common activity.

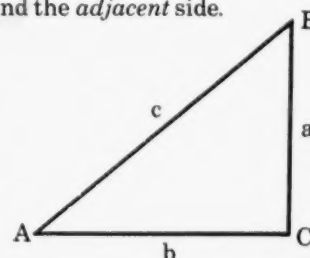
2. Naming Sides in a Right Triangle

The word trigonometry, which is Greek in origin, means the measurement of triangles. To identify certain ratios in trigonometry, one must name the sides in a right triangle.

The *hypotenuse* is the longest side of a right triangle, and is always opposite the right angle. When the letters A, B, and C identify the vertices of a right triangle, the 90° angle is labeled C by convention.

The two remaining sides of a right triangle are called the *legs*. Each leg may be identified by naming its endpoints, or each leg may be labeled with a small letter (the same letter as the angle that is opposite the side). In relation to one of the acute angles in the triangle, the two legs are called the *opposite* side and the *adjacent* side.

The sides of triangle ABC can be named in two ways. Note that the hypotenuse (c) is constant, whereas the names of the legs (a, b) vary according to the acute angle being considered.



In relation to acute $\angle A$:
 the hypotenuse is $\overline{AB}$ or c
 the side opposite $\angle A$ is $\overline{BC}$ or a
 the side adjacent to $\angle A$ is $\overline{AC}$ or b

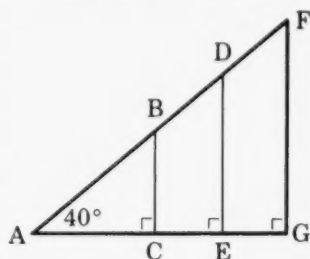
In relation to acute $\angle B$:
 the hypotenuse is $\overline{AB}$ or c
 the side opposite $\angle B$ is $\overline{AC}$ or b
 the side adjacent to $\angle B$ is $\overline{BC}$ or a

Note that $\overline{AB}$ (read as "segment AB") names a segment, while AB (read as "AB") is the measure or the length of the segment.

3. Trigonometric Ratios

Earlier it was noted that the lengths of corresponding sides of similar triangles are in proportion. Before presenting trigonometric ratios, establish the fact that corresponding angles of similar triangles have the same measure.

Point out that whether presented as 3 separate triangles or as overlapping triangles, triangles ABC, ADE, and AFG are similar triangles. Cite the corresponding angles of these triangles, and their measures:



Triangle ABC	Triangle ADE	Triangle AFG
$m\angle A = 40$	$m\angle A = 40$	$m\angle A = 40$
$m\angle C = 90$	$m\angle E = 90$	$m\angle G = 90$
$m\angle B = 50$	$m\angle D = 50$	$m\angle F = 50$

The students should be able to observe a variety of proportions in these similar triangles:

$$\frac{BC}{AC} = \frac{DE}{AE} = \frac{FG}{AG}$$

$$\frac{BC}{AB} = \frac{DE}{AD} = \frac{FG}{AF}$$

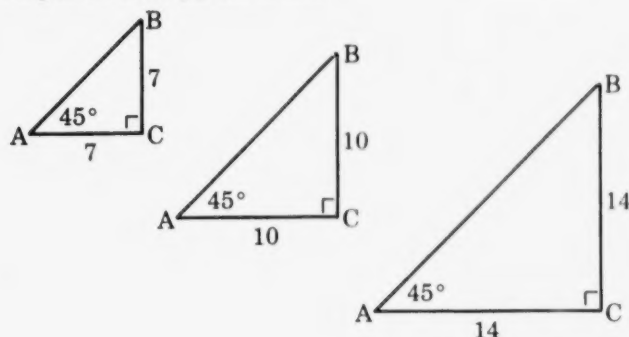
$$\frac{AC}{AB} = \frac{AE}{AD} = \frac{AG}{AF}$$

a. Tangent ratio

Develop the formula for the tangent of an acute angle of a right triangle. Use similar right triangles, and have the students observe the constant ratio for a given angle measure.

Example: For each of the similar triangles, find the ratio of

$\frac{\text{length of side opposite } \angle A}{\text{length of side adjacent to } \angle A}$



$$\text{Answer: } \frac{7}{7} = 1 \quad \frac{10}{10} = 1 \quad \frac{14}{14} = 1$$

No matter how large or small the triangle is, the ratio of "opposite to adjacent" equals the constant value 1 for the given acute angle of 45° . The value of the ratio changes, however, as the measure of the acute angle changes.

In general terms, for any right triangle ABC in which $m\angle C = 90^\circ$:

$$\text{tangent } A = \frac{\text{length of side opposite } \angle A}{\text{length of side adjacent to } \angle A} = \frac{BC}{AC} \text{ or } \frac{a}{b}$$

It is acceptable to abbreviate tangent A as "tan A" and to write:

$$\tan A = \frac{\text{opposite}}{\text{adjacent}}$$

Introduce a *table of tangent values*, as shown on the next page, for acute angles whose measures are given in whole degrees. Show the students how to locate values, such as $\tan 45^\circ = 1$.

Every tangent value in the table, except $\tan 45^\circ$, is a rational approximation to the nearest ten-thousandth. Show how the table is also used to find angle measures, but only to the nearest degree.

Example: If $\tan B = 4.3315$, find the measure of acute $\angle B$.

Answer: 77°

Show how the table and the formula are used to find an unknown length of a side in a right triangle.

Example: In triangle ABC, $m\angle C = 90^\circ$,

$$m\angle A = 38^\circ, \text{ and } AC = 40$$

Find BC to the nearest whole number.

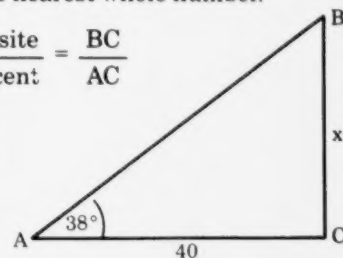
$$\tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{BC}{AC}$$

$$\tan 38^\circ = \frac{x}{40}$$

$$.7813 = \frac{x}{40}$$

$$40(.7813) = x$$

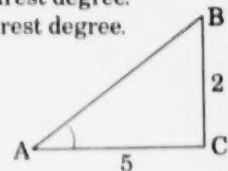
$$31.2520 = x$$



Answer: $BC = 31$ (nearest whole number)

Example: In triangle ABC, $m\angle C = 90^\circ$, $BC = 2$, and $AC = 5$.

- Express $\tan A$ as a decimal to four places.
- Find $m\angle A$ to the nearest degree.
- Find $m\angle B$ to the nearest degree.



Analysis:

$$(a) \tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{2}{5} = .4000$$

(b) Since .4000 is closest to .4040 and $.4040 = \tan 22^\circ$ in the table, then $m\angle A = 22^\circ$ to the nearest degree.

(c) Encourage a variety of approaches here, e.g.

$$180^\circ - (90^\circ + 22^\circ) \quad m\angle A + m\angle B = 90 \quad \tan B = \frac{5}{2}$$

$$180^\circ - 112^\circ \quad 22^\circ + m\angle B = 90 \quad \tan B = 2.5000$$

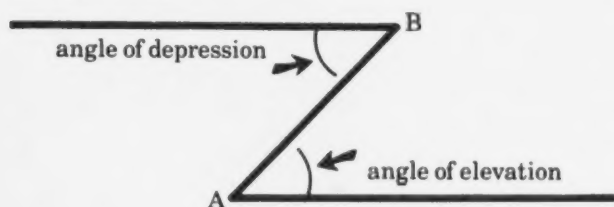
$$68^\circ \quad m\angle B = 90^\circ - 22^\circ \quad m\angle B = 68^\circ$$

Answer: (a) .4000 (b) 22° (c) 68°

Values of Trigonometric Functions

Angle	Sine	Cosine	Tangent	Angle	Sine	Cosine	Tangent
1°	.0175	.9998	.0175	46°	.7193	.6947	1.0355
2°	.0349	.9994	.0349	47°	.7314	.6820	1.0724
3°	.0523	.9986	.0524	48°	.7431	.6691	1.1106
4°	.0698	.9976	.0699	49°	.7547	.6561	1.1504
5°	.0872	.9962	.0875	50°	.7660	.6428	1.1918
6°	.1045	.9945	.1051	51°	.7771	.6293	1.2349
7°	.1219	.9925	.1228	52°	.7880	.6157	1.2799
8°	.1392	.9903	.1405	53°	.7986	.6018	1.3270
9°	.1564	.9877	.1584	54°	.8090	.5878	1.3764
10°	.1736	.9848	.1763	55°	.8192	.5736	1.4281
11°	.1908	.9816	.1944	56°	.8290	.5592	1.4826
12°	.2079	.9781	.2126	57°	.8387	.5446	1.5399
13°	.2250	.9744	.2309	58°	.8480	.5299	1.6003
14°	.2419	.9703	.2493	59°	.8572	.5150	1.6643
15°	.2588	.9659	.2679	60°	.8660	.5000	1.7321
16°	.2756	.9613	.2867	61°	.8746	.4848	1.8040
17°	.2924	.9563	.3057	62°	.8829	.4695	1.8807
18°	.3090	.9511	.3249	63°	.8910	.4540	1.9626
19°	.3256	.9455	.3443	64°	.8988	.4384	2.0503
20°	.3420	.9397	.3640	65°	.9063	.4226	2.1445
21°	.3584	.9336	.3839	66°	.9135	.4067	2.2460
22°	.3746	.9272	.4040	67°	.9205	.3907	2.3559
23°	.3907	.9205	.4245	68°	.9272	.3746	2.4751
24°	.4067	.9135	.4452	69°	.9336	.3584	2.6051
25°	.4226	.9063	.4663	70°	.9397	.3420	2.7475
26°	.4384	.8988	.4877	71°	.9455	.3256	2.9042
27°	.4540	.8910	.5095	72°	.9511	.3090	3.0777
28°	.4695	.8829	.5317	73°	.9563	.2924	3.2709
29°	.4848	.8746	.5543	74°	.9613	.2756	3.4874
30°	.5000	.8660	.5774	75°	.9659	.2588	3.7321
31°	.5150	.8572	.6009	76°	.9703	.2419	4.0108
32°	.5299	.8480	.6249	77°	.9744	.2250	4.3315
33°	.5446	.8387	.6494	78°	.9781	.2079	4.7046
34°	.5592	.8290	.6745	79°	.9816	.1908	5.1446
35°	.5736	.8192	.7002	80°	.9848	.1736	5.6713
36°	.5878	.8090	.7265	81°	.9877	.1564	6.3138
37°	.6018	.7986	.7536	82°	.9903	.1392	7.1154
38°	.6157	.7880	.7813	83°	.9925	.1219	8.1443
39°	.6293	.7771	.8098	84°	.9945	.1045	9.5144
40°	.6428	.7660	.8391	85°	.9962	.0872	11.4301
41°	.6561	.7547	.8693	86°	.9976	.0698	14.3007
42°	.6691	.7431	.9004	87°	.9986	.0523	19.0811
43°	.6820	.7314	.9325	88°	.9994	.0349	28.6363
44°	.6947	.7193	.9657	89°	.9998	.0175	57.2900
45°	.7071	.7071	1.0000	90°	1.0000	.0000	

The concept of indirect measure should be stressed in problem-solving situations, and a variety of practical applications should be explored. Explain angle of elevation and angle of depression.



In the diagram, point B is at a higher altitude than point A, and $\overline{AB}$ is a line of sight, whether we look from A up to B or from B down to A. The *angle of elevation* is an acute angle formed by two rays: the line of sight from A to B, and a horizontal line through A. The *angle of depression* is an acute angle formed by two rays: the line of sight from B to A, and a horizontal line through B. At this stage, a protractor can be used to demonstrate that these angles are equal in measure.

In each application, the students should draw a right triangle, label sides and angles, identify two known quantities, and use the formula to solve for the unknown quantity.

Example: At a point on the ground 20 meters away from the school, the angle of elevation to the top of the building is 50° . Find the height of the school building to the nearest meter.

$$\tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{BC}{AC}$$

$$\tan 50^\circ = \frac{x}{20}$$

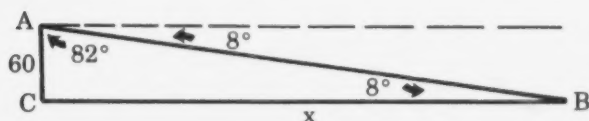
$$1.1918 = \frac{x}{20}$$

$$x = 20(1.1918)$$

$$x = 23.8360$$

Answer: 24 meters

Example: At a beach, from the top of a 60-foot cliff to a boat in the water the angle of depression is 8° . Find to the nearest foot the distance from the bottom of the cliff to the boat. (Note how the angles in the triangle are related to the angle of depression. Show two variations of the tangent ratio.)



$$\tan 8^\circ = \frac{60}{x}$$

$$\text{or } \tan 82^\circ = \frac{x}{60}$$

$$.1405 = \frac{60}{x}$$

$$7.1154 = \frac{x}{60}$$

$$.1405x = 60$$

$$x = 60(7.1154)$$

$$x = \frac{60.0000}{.1405}$$

$$x = 426.9240$$

$$x = 427.0$$

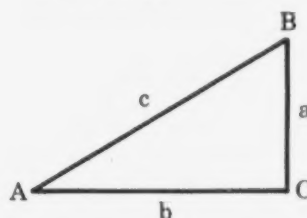
$$x = 427.0$$

Answer: 427 feet

b. Sine ratio

Develop the sine ratio in a manner similar to that of the tangent ratio. In right triangle ABC in which $m\angle C = 90^\circ$:

$$\sin A = \frac{\text{length of side opposite } \angle A}{\text{length of hypotenuse}} = \frac{BC}{AB} = \frac{a}{c}$$



It is common to abbreviate sine A as "sin A" and to write:

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}}$$

With the exception of $\sin 30^\circ = \frac{1}{2} = .5000$, every sine ratio in the table (page 33) is a rational approximation to the nearest ten-thousandth.

Applications involving indirect measures should be studied.

Example: To fly a kite, Stephen let out 240 meters of string. The string made an angle of 63° with the ground. How high, to the nearest meter, is the kite above the ground?

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{BC}{AB}$$

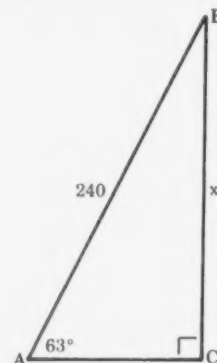
$$\sin 63^\circ = \frac{x}{240}$$

$$.8910 = \frac{x}{240}$$

$$x = 240(.8910)$$

$$x = 213.8400$$

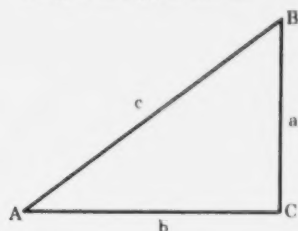
Answer: 214 meters



c. Cosine ratio

A similar plan can be used to develop the cosine ratio. For right triangle ABC in which $m\angle C = 90^\circ$:

$$\cosine A = \frac{\text{length of side adjacent to } \angle A}{\text{length of hypotenuse}} = \frac{AC}{AB} = \frac{b}{c}$$



The abbreviation for cosine A is "cos A," and we write:

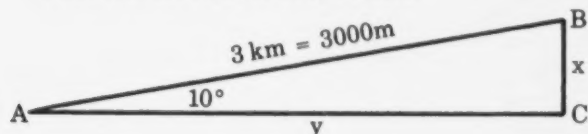
$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}}$$

With the exception of $\cos 60^\circ = \frac{1}{2} = .5000$, every cosine ratio in the table is a rational approximation to the nearest ten-thousandth.

Example: An airplane takes off, rising at an angle of 10° with the ground. Assume that the plane travels an actual distance of 3 kilometers. Find:

- to the nearest ten meters, the height of the plane above the ground,
- to the nearest ten meters, the ground distance covered.

First convert 3 km to 3000 meters.



$$(1) \sin 10^\circ = \frac{x}{3000} \text{ or } \cos 80^\circ = \frac{x}{3000}$$

$$x = 3000(.1736) = 520.8000 = 520$$

$$(2) \cos 10^\circ = \frac{y}{3000} \text{ or } \sin 80^\circ = \frac{y}{3000}$$

$$y = 3000(.9848) = 2954.4000 = 2950$$

Answer: (a) 520 meters (b) 2950 meters

C. PERCENTS

The students should know that a *percent* is a ratio with a denominator of 100. Percent, which means "per hundred" or "hundredths," is represented by the symbol %.

1. Percents Greater than 100% and Less than 1%

Procedures to find common fraction-decimal-percent

equivalents are outlined in *Mathematics 7 Syllabus*. These conversions should be reviewed and extended to include percents greater than 100% and less than 1%. For example:

$$15\% = .15 = \frac{15}{100} \text{ (or } \frac{3}{20} \text{)}$$

$$\text{Multiple by 10: } \begin{array}{ccc} \downarrow & & \downarrow \\ 150\% & = & 1.50 \text{ (or } 1.5) \end{array} \quad \frac{150}{100} \text{ (or } \frac{3}{2} \text{)}$$

Observe that $150\% = 100\% + 50\%$. It can be reasoned that every percent greater than 100% will equal a common fraction or decimal greater than 1.

Example: Complete the table, finding equivalent values for each line.

Common Fraction	Decimal	Percent
$\frac{5}{4}$		
	4.2	
		203%

Answer:

$\frac{5}{4}$	1.25	125%
$\frac{42}{10} = \frac{21}{5}$	4.2	420%
$\frac{203}{100}$	2.03	203%

Percents less than 1% can be introduced by starting with known equivalents and by using various computational schemes. For example:

(a) Using subtraction

$$\begin{array}{rclcl} 12\frac{1}{2}\% & = & 12.5\% & = & .125 \\ - 12\% & = & - 12.0\% & = & - .12 \\ \hline \frac{1}{2}\% & = & 0.5\% & = & .005 \end{array}$$

(b) Using multiplication

$$\frac{1}{2}\% \text{ means } \frac{1}{2} \text{ of } 1\%, \text{ or } \frac{1}{2} \times .01 = .005.$$

(c) Using division

$$1\% \text{ (or } 1.0\%) = .01 \text{ (or } .010) = \frac{1}{100} \text{ (or } \frac{10}{1000})$$

Divide by 2, or multiply by $\frac{1}{2}$:

$$\frac{1}{2}\% \text{ (or } 0.5\%) = .005 = \frac{5}{1000} \text{ (or } \frac{1}{200})$$

Example: Express $\frac{3}{4}\%$ as a decimal and a common fraction in lowest terms.

$$\frac{3}{4}\% = \frac{3}{4} \times .01 = \frac{.03}{4} = \frac{.0300}{4} = .0075$$

$$\frac{3}{4}\% = \frac{3}{4} \times \frac{1}{100} = \frac{3}{400}$$

Answer: .0075 and $\frac{3}{400}$

2. Working with Percents

The students should continue to solve problems by using percents, although no attempt should be made to categorize the problems into three cases. Techniques, including the use of proportions, arithmetic approaches, and the use of equations, are outlined in *Mathematics 7 Syllabus*. Extend these problems to include percents greater than 100% and less than 1%. Although applications with percentage are studied in the next section, some applications are mentioned here.

a. Finding a percent of a number

Example: Find 0.8% of 350.

By a proportion:

$$\frac{x}{350} = \frac{0.8}{100}$$

$$100x = 280.0$$

$$x = 2.8$$

Answer: 2.8

By arithmetic:

$$0.8\% \text{ of } 350$$

$$.008 \times 350$$

$$2.800$$

Example: A store owner bought shirts for \$8.50 each and marked up this price by 80%. The selling price of each shirt is now 100% of the cost plus 80% *mark-up*, or 180%. Find the selling price of a shirt. (What is 180% of \$8.50?)

By a proportion:

$$\frac{x}{8.50} = \frac{180}{100}$$

$$100x = 1530.00$$

$$x = 15.30$$

Answer: \$15.30

By arithmetic:

$$180\% \text{ of } 8.50$$

$$1.8 \times 8.50$$

$$15.300$$

b. Finding what percent one number is of another

Example: 36 is what percent of 30?

By a proportion:

$$\frac{36}{30} = \frac{x}{100}$$

$$30x = 3600$$

$$x = 120$$

By arithmetic:

$$\frac{36}{30} = \frac{6}{5}$$

$$\frac{6}{5} \times 100\% = \frac{600}{5}\% = 120\%$$

Answer: 120%

Example: Mr. Keenan earns \$320 weekly and has a \$2 deduction for medical insurance taken from his weekly pay check. What percent of his earnings is deducted for medical insurance?

By a proportion:

$$\frac{2}{320} = \frac{x}{100}$$

$$320x = 200$$

$$x = \frac{20}{32} = \frac{5}{8}$$

Answer: $\frac{5}{8}\%$ or 0.625%

By arithmetic:

$$\frac{2}{320} = \frac{1}{160}$$

$$\frac{1}{160} \times 100\% = \frac{100}{160}\% = \frac{5}{8}\%$$

c. Finding a number when a percent of it is given

Example: If 7 is $\frac{1}{3}\%$ of a number, what is the number?

By a proportion:

$$\frac{7}{x} = \frac{\frac{1}{3}}{100}$$

$$\frac{1}{3}x = 700$$

$$x = 2100$$

Answer: 2100

By arithmetic:

$$\frac{1}{3}\% \text{ is } 7$$

multiply by 3

$$1\% \text{ is } 21$$

multiply by 100

$$100\% \text{ is } 2100$$

Example: A share of stock now worth \$35 equals 250% of the value at which it was bought. Find the original price of one share of stock.

By a proportion:

$$\frac{35}{x} = \frac{250}{100}$$

$$250x = 3500$$

$$x = 14$$

Answer: \$14

By arithmetic:

$$250\% \text{ is } 35$$

divide by 5

$$50\% \text{ is } 7$$

multiply by 2

$$100\% \text{ is } 14$$

3. Problem Solving

Percent should be applied to a variety of problem-solving situations. Percent of increase/decrease and compound interest are introduced at this grade level. Although commission, discount, and taxes are to be reviewed through some problems, a broader range of applications should be explored, using such sources as the daily newspaper and the almanac. Circle graphs may be integrated with this unit or delayed until the Geometry strand.

a. Percent of increase and percent of decrease

Some student judgment is required in determining the base, or the number upon which the percent is taken. Suggested applications include profit and loss, population changes, markups and discounts, stocks, and any data in which fluctuations or changes may occur.

Example: In one year, Nadia's weight increased from 60 kg to 66 kg. By dieting during the following year, Nadia brought her weight back down to 60 kg.

- Find the percent of increase in Nadia's weight for the first year.
- Find the percent of decrease in her weight for the second year.

- Gain = 6
Base or start = 60
- Loss = 6
Base or start = 66

$$\frac{6}{60} = \frac{x}{100}$$

$$\frac{6}{66} = \frac{x}{100}$$

Answer: (a) 10% increase (b) $9\frac{1}{11}\%$ decrease

Example: The ABC Company reported that profits for 1984 declined 18% from the 1983 figures. If the profit for 1983 was \$750,000, what was the profit for 1984?

Answer: \$615,000

(Did any student solve the problem by finding 82% of 750,000?)

Example: (Guess this answer before performing the computations.) A share of stock costs 50 dollars. Under which set of conditions would the stock have the greatest value?

- The stock decreases 20% in value, then increases 20%.
- The stock increases 20% in value, then decreases 20%.
- The stock does not change in value.

Answer: (c) 50 dollars is the greatest value. (Under both choices (a) and (b), the resulting share is worth 48 dollars.)

b. Compound interest

Review the formula for simple interest ($I = PRT$), and note that its importance is limited to "estimating" the actual interest earned or paid.

Example: In his bank account, Jack kept \$2,000 at 8% interest for 1 year. Estimate the simple interest earned and the new total after 1 year.

$$I = PRT = \$2000 \times 8\% \times 1 = \$160$$

$$\text{Amount } A = P + I = \$2000 + \$160 = \$2160$$

Answer: Interest = \$160, and new principal = \$2160. To develop the concept of compound interest, redo the problem, but show what happens if interest is compounded semiannually (twice each year):

First 6 months	Second 6 months
$I = PRT$	$I = PRT$
$I = \$2000 \times 8\% \times \frac{1}{2}$	$I = \$2080 \times 8\% \times \frac{1}{2}$
$I = \$80$	$I = \$83.20$
Principal after 6 months $A = P + I$	Principal after full year $A = P + I$
$A = \$200 + 80 = \$2080$	$A = \$2080 + 83.20 = \$2163.20$

The interest compounded quarterly (four times each year) is greater than the interest compounded semiannually, which is greater than the simple or estimated interest. Compare these results:

Principal	Rate	Interest	Amount After 1 Year
\$2000	8% yearly	\$160	\$2160
\$2000	8%, compd. semiannually	\$163.20	\$2163.20
\$2000	8%, compd. quarterly	\$164.86	\$2164.86

In most banks, interest is compounded on a daily basis. Mention "Day of Deposit to Day of Withdrawal" accounts.

Once the basic concept of compound interest is understood, explain that these problems are handled more realistically by using the compound interest formula or compound interest tables.

Optional: the compound interest formula, which is

$$A = P(1 + r)^n$$

where P = principal at start

r = interest rate per period

$$= \frac{\text{annual rate } R \text{ in decimal form}}{\text{number of periods per year}}$$

n = number of compounding periods

A = final amount, or principal after full interest term

Demonstrate how the formula applies to the cases studied earlier. Use a calculator to perform the computation.

Simple

$$\text{interest: } A = 2000(1 + .08)^1 = 2000(1.08) = 2160$$

Semiannual

$$\text{compounding: } A = 2000\left(1 + \frac{.08}{2}\right)^2 = 2000(1.04)^2 = 2163.20$$

$$\text{Quarterly: } A = 2000\left(1 + \frac{.08}{4}\right)^4 = 2000(1.02)^4 = 2164.86$$

Example: A \$300 balance is owed on a credit card at 18% interest compounded monthly. What amount must be paid back in 2 months to clear this debt?

$$A = P(1 + r)^n$$

$$A = 300(1 + \frac{.18}{12})^2 = 300(1.015)^2 = 309.0675$$

Answer: \$309.07

Project: Write a computer program to generate compound interest.

c. Applications from print media

Significant applications with percent can be found in newspapers. These suggestions are neither mandatory nor restrictive: the results of surveys and polls, sports statistics, data from financial pages, advertisements of sales, etc. The almanac is another rich source of problems.

Example: Every year, the prices of the same items are compared to the prices during a base year, 1967. These numbers are known as the Consumer Price Index (C.P.I.).

YEAR	1965	1966	1967	1968	1969
C.P.I.	94.5	97.2	100.0	104.2	109.8

- (1) If it cost \$100 to buy certain products and services in 1967, what did these same items cost in 1965?
- (2) Find the percent of increase in the C.P.I. from 1967 to 1969.
- (3) Find the percent of decrease in the C.P.I. from 1967 to 1966.
- (4) Estimate the C.P.I. for a series of years starting in 1970. Then compare your estimates with the actual indices by checking the proper references.

Answer: (1) \$94.50 (2) 9.8% (3) 2.8% (4) varies

d. Circle graphs

In addition to reading circle graphs, the students should use percent to construct these graphs. Inasmuch as protractors are used in this activity, data should be restricted to those cases where the degree measure of each central angle is an integer. This topic may be presented here or in the Statistics unit.

Example: A survey of the TV networks the students

watched at 8 p.m. Sunday evening revealed these statistics:

ABC (6); CBS (7); NBC (3); PBS (5); no TV (9)

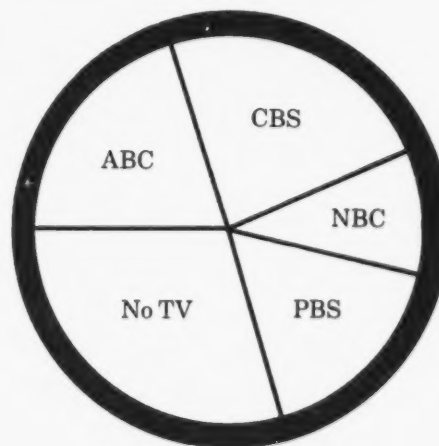
- (1) Using the data shown above, for each category find: fraction of total, percent of total, degree measure for circle graph.
- (2) Using a protractor and a ruler, draw a circle graph of the data.

Answer:

(1)

Category	Number	Fraction of Total	% of Total	Degree Measure
ABC	6	$6/30 = 1/5$	20%	72°
CBS	7	$7/30$	$23\frac{1}{3}\%$	84°
NBC	3	$3/30 = 1/10$	10%	36°
PBS	5	$5/30 = 1/6$	$16\frac{2}{3}\%$	60°
no TV	9	$9/30 = 3/10$	30%	108°
Totals	30	$30/30 = 1$	100%	360°

(2) TV Networks Viewed, 8 p.m. Sunday



Finally, interpret information from the circle graph.

IV. PROBABILITY AND STATISTICS

It is advisable to begin this unit by reviewing some basic definitions and rules about probability encountered in grade 7. In grade 8, the Counting Principle is extended to three or more tasks, permutations are developed, mutually exclusive events are defined, and probability is applied to these concepts.

For statistics, in grade 8 the measures of central tendency are extended to include grouped data using whole number intervals, and the frequency polygon is introduced.

A. PROBABILITY

The rule for *theoretical probability* is given as $P(E) = f/n$, or: the probability of an event =

$$\frac{\text{The number of favorable outcomes}}{\text{Total number of possible outcomes}}$$

Some students may have seen a set theory approach to probability and may use the rule $P(E) = \frac{n(E)}{n(S)}$. Here,

$n(E)$ represents the number of outcomes in event E, and $n(S)$ represents the total number of outcomes in the sample space S.

1. Review Theoretical Probability

A review of theoretical probability should occur as a by-product of problem solving, not as the presentation of a glossary of terms. Concepts to discuss include terms such as sample space and event, the rule for theoretical probability, and the range of probabilities. A set theory approach to probability is given within this *Syllabus*, but it is not mandated; the approach is optional for the teacher and/or the students.

The students should know that the *sample space* is a listing or set of all possible outcomes, and an *event* is a listing of favorable outcomes. In set theory, an event is a subset of the sample space.

Example: Terri wrote the numbers 4, 8, 12, 16, and 20 on five separate pieces of paper and placed them in a box. Martin selected one of these numbers at random. Find the probability that the number Martin selected is (a) a multiple of 8, (b) a multiple of 2, (c) a multiple of 7, (d) not a multiple of 3.

Analysis:

- (a) Sample space:
Event A (multiple of 8):

$$P(A) = \frac{f}{n} = \frac{2}{5}$$

- (b) Sample space:
Event B (multiple of 2):

$$P(B) = \frac{f}{n} = \frac{5}{5} = 1$$

- (c) Sample space:
Event C (multiple of 7):

$$P(C) = \frac{f}{n} = \frac{0}{5} = 0$$

The probability of an impossibility is 0.

- (d) Sample space:
Event D (not a multiple of 3):

$$P(D) = \frac{f}{n} = \frac{4}{5}$$

Without Sets:

- 4, 8, 12, 16, 20
8, 16

- 4, 8, 12, 16, 20
4, 8, 12, 16, 20

- 4, 8, 12, 16, 20
none

- 4, 8, 12, 16, 20
4, 8, 16, 20

Set Approach:

- $S = \{4, 8, 12, 16, 20\}$
 $A = \{8, 16\}$

- $S = \{4, 8, 12, 16, 20\}$
 $B = \{4, 8, 12, 16, 20\}$

- $S = \{4, 8, 12, 16, 20\}$
 $C = \{ \}$

- $S = \{4, 8, 12, 16, 20\}$
 $D = \{4, 8, 16, 20\}$

$$P(A) = \frac{n(A)}{n(S)} = \frac{2}{5}$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{5}{5} = 1$$

$$P(C) = \frac{n(C)}{n(S)} = \frac{0}{5} = 0$$

$$P(D) = \frac{n(D)}{n(S)} = \frac{4}{5}$$

As an alternative approach, find $P(E)$ or the probability that the number is a multiple of 3. Then subtract this probability from 1. $P(\text{not multiple of 3}) = 1 - P(\text{multiple of 3})$

$$= 1 - \frac{1}{5} = \frac{4}{5}$$

As an alternate approach with sets, use the *complement*. Let $P(E) = P(\text{multiple of 3})$.

$$P(\bar{E}) = 1 - P(E) = 1 - \frac{1}{5} = \frac{4}{5}$$

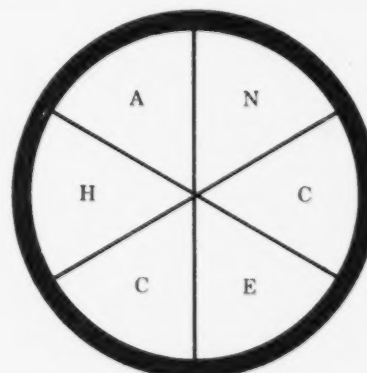
Each of the examples cited above illustrates that the probability of any event is a value from 0 to 1, or:

$$0 \leq P(E) \leq 1$$

$$\text{Answer: (a) } \frac{2}{5} \quad (b) 1 \quad (c) 0 \quad (d) \frac{4}{5}$$

Example: In a game of CHANCE, the spinner is divided into 6 regions, each equally likely to occur when the arrow is spun.

- (a) Find two events, R and T, in which $P(R) = 2P(T)$.
(b) Find two events, X and Y, in which $P(X) = P(Y)$.



Answer: Many responses are possible. For example:

- (a) Let R = spin a "C" and let T = spin an "N."
Or, let R = spin a consonant and let T = spin a vowel.
- (b) Let X = spin an "E" and let Y = spin an "A."
Or, let X = spin a letter from "HEN" and let Y = spin "A" or "C."

The example just given can lead to a discussion of fair games. Recall fair vs. biased objects. Inasmuch as each of the 6 regions is equally likely to occur, the spinner is a *fair* object. However, from a viewpoint of the letters shown in the regions, because "C" has a greater probability of occurring, the spinner is *biased* in favor of the letter "C." Challenge the students to create their own games, using spinners, cards, or dice, and then to determine which games are fair and which are not.

2. Counting Procedures related to Probability

a. Independent events

Independent events are two or more events in which each event has no influence over the others. For example, in getting a tail on the toss of one coin, a tail on the toss of a second coin, and a tail on the toss of a third coin are independent events. In fact, the result of each new toss of the *same* fair coin, no matter how often it is flipped, is independent of all other tosses.

b. Extending the Counting Principle

The *Counting Principle* states that if one task can be done in M ways and another in N ways, there are $M \times N$ ways to do the two tasks, usually in the order given. This principle is now extended to three or more tasks.

Example: How many different results are possible from 3 tosses of a coin? Analysis: Each toss has 2 possible outcomes, H or T. By the Counting Principle, there are $2 \times 2 \times 2$ possible results.

Answer: 8

Example: Len has 9 shirts, 5 pairs of pants, and 3 ties. How many possible outfits of 1 shirt, 1 pair of pants, and 1 tie could Len create?

$$9 \times 5 \times 3 = 135$$

Answer: 135 outfits

Example: The school cafeteria offers 3 main courses, 5 types of sandwiches, 4 different beverages, and 6 kinds of dessert. How many possible lunches could be formed consisting of 1 main course or sandwich, 1 beverage, and 1 dessert?

$$\text{First, } 3 + 5 = 8. \text{ Then, } 8 \times 4 \times 6 = 192$$

Answer: 192 lunches

3. Permutations

(In K-6, permutations were studied, but they were identified more often as arrangements. No formulas were stated in these grades, nor was any mention made of factorial notation).

a. Definition

Permutations are the possible arrangements that can be made with a set of objects, where each different order is considered a different arrangement.

Example: To answer three problems at the board, the teacher calls on three students: Hernando, Olga, and Tom. How many possible ways can the teacher assign these students to answer one problem each?

There are 6 possible permutations (arrangements), where the order given is assumed to be problems 1, 2, 3, and each student is identified by an initial:

H,O,T	O,H,T	T,H,O
H,T,O	O,T,H	T,O,H

Answer: 6

Example: How many different permutations of the letters in the word HOT are possible?

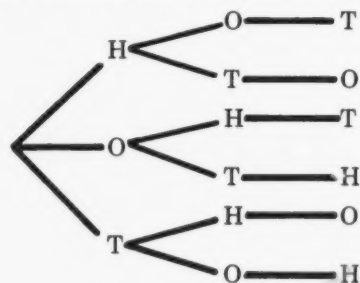
HOT	OHT	THO
HTO	OTH	TOH

Answer: 6

(Note that only three-letter arrangements were used, and no letter was used more than once in an arrangement. This question may also be written as, "How many three-letter words can be formed with one each of the letters H, O, and T?" Here, any arrangement of letters is defined to be a word, including nonsense words.)

Both questions just presented could be answered by expanding the Counting Principle to three activities. Think of three blanks, , and the ways in which they can be filled. The first can be filled with any of the three letters H, O, and T. Once that is selected, we have two choices for the second letter. Finally the third blank is filled in with the remaining letter.

If a tree diagram is drawn, six paths can be followed.



In brief, $3 \times 2 \times 1 = 6$ permutations

b. Factorial notation

A short way to write the product $3 \times 2 \times 1$ is $3!$, read as "three factorial." Similarly, $4!$ means $4 \times 3 \times 2 \times 1$, or 24. In general, for any counting number n :

$$n! = n \times (n-1) \times (n-2) \times \dots \times 1$$

$$n! = n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1$$

Answers to problems may be left in factorial notation when computation becomes cumbersome, as in the following example.

Example: In how many ways can the teacher assign 30 students to the 30 desks in a room?

Answer: $30!$ (which is $30 \times 29 \times 28 \times \dots \times 3 \times 2 \times 1$)

c. n objects taken n at a time

Exercises which deal with 30 students assigned to 30 desks, or 4 students assigned to answer 4 questions, are illustrations of "permutations of n objects taken n at a time." This is written in symbols as ${}_nP_n$ where:

$${}_nP_n = n! = n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1$$

Example: How many different 4-digit numbers can be made by using the digits 3, 5, 7, and 9 if each digit appears only once in each number?

This is a permutation of 4 objects taken 4 at a time.

Therefore, ${}_4P_4 = 4! = 4 \times 3 \times 2 \times 1 = 24$.

Answer: 24

After the students solve this problem by using the formula, have them identify these 24 numbers, either with a tree diagram or by a straightforward listing.

d. n objects taken r at a time

At times, problems may involve permutations of " n " objects where "fewer than n " objects are used in each arrangement. The students should apply problem-solving techniques to answer such exercises before a rule is presented to them. Such a rule is **optional**.

Example: There are 5 students in the first row of a classroom: Alice, Ben, Chris, Doris, and Elmer. In how many different ways can the teacher assign one of these students to erase the board and a second of these students to collect homework papers?

Solution A: Some students may list all possible arrangements, identifying each student by an initial.

A,B	B,A	C,A	D,A	E,A
A,C	B,C	C,B	D,B	E,B
A,D	B,D	C,D	D,C	E,C
A,E	B,E	C,E	D,E	E,D

Solution B: Some students may use the Counting Principle. Any of 5 students can be assigned to erase the board. Then any of 4 students can be assigned to collect papers. There are 5×4 , or 20 permutations.

Answer: 20

The example just presented involved "a permutation of 5 students taken 2 at a time." In symbols, this is ${}_5P_2$. The teacher may rely on the Counting Principle or may present either one or both of the following rules to answer such questions.

Rule 1: To evaluate ${}_nP_r$, a "permutation of n objects taken r at a time," begin to write the factors of $n!$ but stop upon reaching r factors.

$${}_5P_2 = 5 \cdot 4 = 20$$

(2 factors)

Rule 2: To evaluate ${}_nP_r$, a "permutation of n objects taken r at a time," divide $n!$ by $(n-r)!$.

$${}_5P_2 = \frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 20$$

Example: There are 30 students in a class. In how many different ways can the class elect a president, a treasurer, and a secretary?

Solution A: By the Counting Principle,

$$30 \times 29 \times 28 = 24,360$$

Solution B: ${}_{30}P_3 = 30 \cdot 29 \cdot 28 = 24,360$
(3 factors)

Solution C:

$${}_{30}P_3 = \frac{30!}{(30-3)!} = \frac{30!}{27!} = \frac{30 \cdot 29 \cdot 28 \cdot \cancel{27} \cdot \cancel{26} \cdot \cancel{25} \dots 1}{\cancel{27} \cdot \cancel{26} \cdot \cancel{25} \dots 1}$$

Answer: 24,360

After studying permutations, the students should be exposed to exercise sets that provide a mix of problems. Some problems should contain permutations, while others which do not involve permutations should be solved by the Counting Principle. Consider the following example.

Example: A lottery ticket contains a 4-digit number. Find the total possible amount of 4-digit numbers when:
(a) a digit may appear only once in a number. (b) digits may appear more than once in a number.

(a) This is a permutation of 10 digits taken 4 at a time.

$${}_{10}P_4 = 10 \cdot 9 \cdot 8 \cdot 7 = 5,040$$

(b) Any of 10 digits may appear in each place. By the Counting Principle:

$$10 \cdot 10 \cdot 10 \cdot 10 = 10,000 \text{ numbers}$$

Answer: (a) 5,040 (b) 10,000

Optional: The topic of combinations may arise in the classroom because students were exposed to this topic in

earlier grades. Since combinations will be studied in later years, it is recommended that discussion of this topic be kept to a minimum level here.

4. Probability of Two or More Events

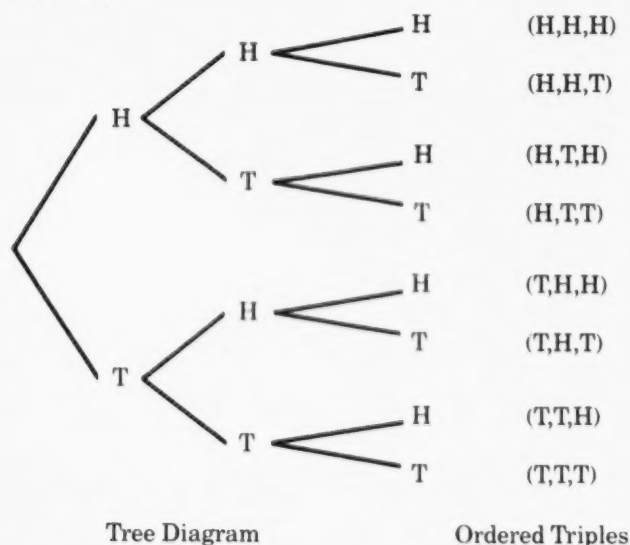
The key to finding the probability of an event, no matter how complicated, lies in knowing two facts: the number of elements in the sample space, and the number of elements in the event. A sample space may sometimes be a permutation.

a. Sample space for two or more independent events

A sample space of two or more independent events is usually displayed as a tree diagram or as a listing of ordered n-tuples (that is, ordered pairs or ordered triples, etc.).

Example: Indicate all possible outcomes when a fair coin is tossed 3 times.

Answer: Recall that each coin has 2 outcomes, H or T, and by the Counting Principle there are $2 \times 2 \times 2$ or 8 outcomes.



b. Probability of two or more independent events

There are two approaches that can be taken when solving problems involving probabilities and permutations, as shown in the following examples.

Example: If 3 fair coins are tossed, find the probability that all are heads.

Solution A: Write the sample space and use the rule for theoretical probability:

$$P(3 \text{ heads}) = \frac{f}{n} = \frac{1}{8}$$

Solution B: For independent events, the probability of each occurring is the product of their independent probabilities.

$$\begin{aligned} P(3 \text{ heads}) &= P(\text{head}) \times P(\text{head}) \times P(\text{head}) \\ &= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \\ &= \frac{1}{8} \end{aligned}$$

Answer: $\frac{1}{8}$

c. Probability of two mutually exclusive events

Two events are *mutually exclusive* if it is impossible for them to occur at the same time. The probability of two mutually exclusive events occurring is equal to the sum of their probabilities:

$$P(A \text{ or } B) = P(A) + P(B)$$

In terms of sets, A and B are disjoint. That is, the intersection of A and B is empty.

Example: If a fair coin is tossed 3 times, what is the probability of getting all heads or all tails?

$$\begin{aligned} P(3 \text{ heads or } 3 \text{ tails}) &= P(3 \text{ heads}) + P(3 \text{ tails}) \\ &= \frac{1}{8} + \frac{1}{8} \\ &= \frac{2}{8} \end{aligned}$$

Answer: $\frac{2}{8}$ or $\frac{1}{4}$

d. Applications

Example: In a hat, there are 5 disks, numbers 1,2,3,4,5. One disk is picked at random, then a second disk is picked at random, to form a two-digit number. What is the probability that both digits are odd?

Solution A: Form the sample space of permutations and apply the rule for probability.

12	21	31	41	51
13	23	32	42	52
14	24	34	43	53
15	25	35	44	54

$$P(\text{both are odd}) = \frac{6}{20}$$

Solution B: Use the product of two probabilities. On the first draw, 3 of 5 disks are odd. Assume an odd number was selected. Then, on the second draw, there are 2 odd numbers out of the 4 disks left.

$$\begin{aligned} P(\text{both are odd}) &= P(\text{first is odd}) \cdot P(\text{second is odd}) \\ &= \frac{3}{5} \cdot \frac{2}{4} = \frac{6}{20} \end{aligned}$$

Answer: $\frac{6}{20}$

Example: In a hat, there are 5 disks, numbered 1,2,3,4,5. Two disks are picked at random to form a two-digit number. What is the probability that the number is a multiple of 5?

Solution A: Use the sample space. Since 4 of the 20 numbers have a units digit of 5,

$$P(\text{multiple of } 5) = \frac{4}{20}$$

Solution B: Use the product of two probabilities. On the first draw, you wish to select any digit except 5. Thus, 4 of the 5 disks are acceptable. On the second draw, you wish to select the digit 5. Thus, only 1 of the 4 remaining disks is acceptable.

$$P(\text{multiple of } 5) = P(\text{any digit except } 5) \cdot P(\text{the digit } 5)$$

$$\frac{4}{5} \cdot \frac{1}{4} = \frac{4}{20}$$

Answer: $\frac{4}{20}$

Example: Jim has 6 candies in his pocket: 4 orange and 2 grape. He takes one candy at random and, after the candy is eaten, he takes and eats a second piece of candy at random. Find the probability that: (a) both candies are grape, (b) both are orange, (c) the flavors are different.

(a) $P(\text{both grape}) = P(\text{first is grape}) \cdot P(\text{second is grape})$

$$\frac{2}{6} \cdot \frac{1}{5} = \frac{2}{30}$$

(b) $P(\text{both orange}) = P(\text{first is orange}) \cdot P(\text{second is orange})$

$$\frac{4}{6} \cdot \frac{3}{5} = \frac{12}{30}$$

(c) $P(\text{different flavors}) = 1 - P(\text{same flavor})$

$$= 1 - P(\text{both grape or both orange})$$

$$= 1 - (\frac{2}{30} + \frac{12}{30})$$

$$= 1 - \frac{14}{30} = \frac{16}{30}$$

Alternative approach to part c: Find the probability of grape first, and orange second. Then find the probability of orange first, and grape second. Add these two probabilities.

$$P(\text{different flavors}) =$$

$$P(\text{grape}) \cdot P(\text{orange}) + P(\text{orange}) \cdot P(\text{grape})$$

$$\frac{2}{6} \cdot \frac{4}{5} + \frac{4}{6} \cdot \frac{2}{5} =$$

$$\frac{8}{30} + \frac{8}{30} = \frac{16}{30}$$

Answer: (a) $\frac{2}{30}$ (b) $\frac{12}{30}$ (c) $\frac{16}{30}$

The example just presented is a variation of the classic problem of drawing colored marbles from an urn. To solve the above problem using a sample space approach, *subscripts* are used to distinguish the individual pieces of candy:

$$G_1, G_2, O_1, O_2, O_3, \text{ and } O_4.$$

Example: An urn contains exactly 5 marbles: 3 red, 1 white, and 1 blue. A marble is drawn at random and not replaced in the urn. A second marble is then drawn at random. Find the probability that: (a) both marbles are red (b) both marbles are white (c) neither is red (d) neither is green (e) at least 1 red marble is drawn.

Solution A: The sample space is listed, using subscripts to distinguish the red marbles (R_1, R_2, R_3), as follows:

$R_1 R_2$	$R_2 R_1$	$R_3 R_1$	$W R_1$	$B R_1$
$R_1 R_3$	$R_2 R_3$	$R_3 R_2$	$W R_2$	$B R_2$
$R_1 W$	$R_2 W$	$R_3 W$	$W R_3$	$B R_3$
$R_1 B$	$R_2 B$	$R_3 B$	$W B$	$B W$

Outcomes are counted for each of the events named, and the basic rule of probability is applied to the event.

(a) $P(\text{both red}) = P(\text{first is red}) \cdot P(\text{second is red})$

$$= \frac{3}{5} \cdot \frac{2}{4} = \frac{6}{20}$$

(b) $P(\text{both white}) = P(\text{first is white}) \cdot P(\text{second is white})$

$$= \frac{1}{6} \cdot \frac{0}{4} = 0$$

(c) $P(\text{neither is red}) = P(\text{not red}) \cdot P(\text{not red})$

$$= \frac{2}{5} \cdot \frac{1}{4} = \frac{2}{20}$$

(d) $P(\text{neither is green}) = P(\text{not green}) \cdot P(\text{not green})$

$$= \frac{5}{5} \cdot \frac{4}{4} = \frac{20}{20} = 1$$

(e) $P(\text{at least 1 red}) = 1 - P(\text{neither is red})$

$$= 1 - \frac{2}{20} = \frac{18}{20}$$

Alternative to part e: If at least one red is drawn, then either one red or two reds are drawn. There are three situations to consider.

$$P(\text{at least 1 red}) = P(\text{red}) \times P(\text{red}) + P(\text{red}) \times P(\text{not red}) + P(\text{not red}) \times P(\text{red})$$

$$= \frac{3}{5} \times \frac{2}{4} + \frac{3}{5} \times \frac{2}{4} + \frac{2}{5} \times \frac{3}{4}$$

$$= \frac{6}{20} + \frac{6}{20} + \frac{6}{20}$$

$$= \frac{18}{20}$$

Answer: $\frac{18}{20}$

B. STATISTICS

In grade 7, the students grouped data into intervals and presented the data in a histogram. In grade 8, the measures of central tendency, previously studied with only small groups of data, will be applied to frequency tables. It is recommended, however, that the intervals in this grouped data be limited to a length of one (that is, a single number), as opposed to an interval spanning a range of numbers.

The examples in this guide are merely illustrative. Statistics becomes more meaningful when the students select surveys and conduct experiments of their own choice.

1. Grouped Data Using Intervals of Length One

The transition from a set of individual scores to a set of grouped data is relatively easy to grasp. However, the students should find each of the measures of central tendency for the ungrouped data before studying procedures with grouped data.

a. The mean

Consider this set of data, showing the ages of 10 students in a junior high school:

13, 12, 12, 14, 12, 13, 12, 13, 12, 13

The *mean* or numerical average is the sum of N numbers, divided by N . In this example:

$$\text{Mean} = \frac{13 + 12 + 12 + 14 + 12 + 13 + 12 + 13 + 12 + 13}{10} = \frac{126}{10} = 12.6$$

The data can be grouped into three intervals, namely the ages 12, 13, and 14. The frequency or number of scores is recorded for each interval. Three products are taken using the formula "Interval $\times$ Frequency," and two columns are added. Notice that the sum of the frequencies is 10 (the N shown previously), and the sum of the products in the third column is 126 (the sum of all ages shown previously).

Interval	Frequency	Interval $\times$ Frequency	
14	1	$14 \times 1 =$	14
13	4	$13 \times 4 =$	52
12	5	$12 \times 5 =$	60
$N = 10$		Total =	126

Using the grouped data:

$$\text{Mean} = \frac{\text{Total}}{N} = \frac{126}{10} = 12.6$$

b. The median

The *median* is the "middle score" for a set of data taken only when the data are arranged in numerical order (from highest to lowest, or from lowest to highest). The median of 1, 2, 7, 8, 9 is 7, the middle score.

If there are two middle numbers, the median is the mean (average) of these two numbers. The 10 ages reported earlier are arranged in numerical order as follows:

12, 12, 12, 12, 12, 13, 13, 13, 13, 14

Since there are two middle numbers, 12 and 13, the median is the mean (average) of these two numbers.

$$\text{Median} = \frac{12 + 13}{2} = \frac{25}{2} = 12.5$$

In the grouped data below, arrows indicate a point at which scores lie above that point, and scores lie below it. That point is the median. Inasmuch as the median falls exactly between the intervals of 12 and 13, take the mean (average) of the two numbers.

Interval	Frequency
14	1
13	4
12	5



$$\text{Median} = \frac{12 + 13}{2} = \frac{25}{2} = 12.5$$

At times, the median will lie somewhere in an interval. In these cases, the median = the interval value.

c. The mode

The *mode* is the number that appears most frequently in a set of data. Some sets of data have more than one mode, while others have no mode. Consider these stipulations to find the mode.

- (1) If one score appears most often, that score is the mode.
- (2) If two or more scores appear the same number of times, and this is more than any other score, then each of these scores is a mode.
- (3) If no score occurs more than once, or if every score appears an equal number of times, there is no mode.

In the ungrouped data 12, 12, 12, 12, 12, 13, 13, 13, 13, 14, the mode = 12, the score appearing most often.

Interval	Frequency
14	1
13	4
12	5

In the grouped data at the left, the highest frequency for any interval is 5. Therefore, the mode = 12, the interval value for that frequency.

In addition to preparing their own statistical studies, the students should have practice in handling the measures of central tendency with small frequency tables. Discuss each set of findings.

Example: A store owner kept a record of the size of ladies' junior dresses sold last week. For these data, find:

- the total frequency
- the mean
- the median
- the mode

Dress Size (Interval)	Number Sold (Frequency)
13	8
11	5
9	11
7	10
5	9
3	7

Solution:

Interval	Frequency	Interval × Frequency
13	8	$13 \times 8 = 104$
11	5	$11 \times 5 = 55$
9	11	$9 \times 11 = 99$
7	10	$7 \times 10 = 70$
5	9	$5 \times 9 = 45$
3	7	$3 \times 7 = 21$
-	N = 50	TOTAL = 394

$$\text{Mean} = 394/50 = 7.88$$

Median lies somewhere in the interval 7.

Mode = 9, since this interval has the greatest frequency.

Answer: (a) 394 (b) 7.88 (c) 7 (d) 9

In the example just given, which measures of central tendency are helpful to the store owner, and which are not? If the information is helpful, what does it tell the store owner?

The mean of 7.88 appears to be of no use in this situation.

The median of 7 tells the store owner that of the dresses sold, half were size 7 or larger, and half were size 7 or smaller.

The mode of 9 tells the store owner that size 9 was the most popular size sold.

In other examples, the mean may be of use, and the mode is unimportant. Vary the examples with small sets of grouped data to reflect realistic situations (salaries, car mileage, sports statistics, etc.) and, where possible, present data so that each of the measures of central tendency will take a turn at being of some use.

2. Presentation of Organized Data

a. Histogram

In grade 7, the students constructed *histograms* to display sets of organized data. Now extend graphing techniques to include constructing frequency polygons.

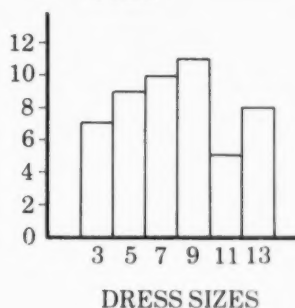
b. Frequency polygon

A *frequency polygon* is a line graph representing the frequency distribution of a given set of data. Every graph should be accompanied by appropriate vertical and horizontal scales, labels, and a title.

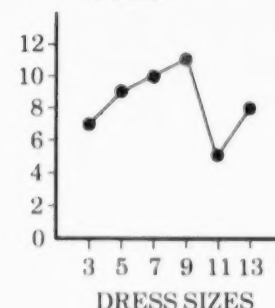
Example: Graph the data collected for the dress sizes sold in the previous problem, using

- a histogram
- a frequency polygon

(a) JUNIOR DRESSES SOLD



(b) JUNIOR DRESSES SOLD



V. ALGEBRA AND COORDINATE GEOMETRY

This unit of Algebra and Coordinate Geometry is an extension of that in grade 7. Emphasis is again placed on translating words to the language and symbols of algebra. Although algebraic techniques may have been applied previously to verbal problems of this curriculum, a formal study of algebraic procedures is now presented in this unit. The scope of this formalization is limited to simplifying and solving first-degree equations and inequalities in one variable. Visual displays are studied through number line graphs.

First-degree equations and inequalities in two variables are presented on a more informal basis and graphed in the coordinate plane.

A. ALGEBRAIC EXPRESSIONS AND SENTENCES

1. Forming Algebraic Expressions and Sentences

The students should know that a sentence contains a complete thought and that an expression or phrase is only part of a sentence. In algebraic terms, an *open sentence* contains one or more variables and states a relation that is either an equation or an inequality. An *algebraic expression* is simply an open expression containing one or more variables.

a. Review of terminology

The concepts, terminology, and definitions given in *Mathematics 7 Syllabus* should be reviewed as the students analyze verbal problems.

Example: The average weight of a passenger in an elevator is estimated to be 150 pounds.

- (a) Represent the weight of n passengers.
- (b) Estimate the weight of 7 passengers.

Answer: (a) $150n$ (b) 1050 pounds

Discuss and define:

the *variable* n
the *constant* 150
the *algebraic expression* or *phrase* $150n$
the *numerical expression* 150×7 , or 1050

Example: If the total weight of the passengers in the above elevator is estimated to be 1200 pounds, how many passengers are in the elevator?

Answer: 8 people

Discuss and define:

the *open sentence* $150n = 1200$, which is an *equation*
the *domain* = $\{1, 2, 3, 4, \dots\}$
false sentences, such as $150 \times 6 = 1200$
the *true sentence* $150 \times 8 = 1200$
the *solution set* $\{8\}$

Set notation is sometimes used as an alternative form in indicating the solution to an equation or an inequality.

Example: For the domain or *replacement set* of counting numbers, solve (a) $x - 3 = 2$ (b) $x - 3 < 2$ (c) $x + 3 = 2$

Answer: (a) $\{5\}$ (b) $\{1, 2, 3, 4\}$ (c) $\{ \}$ or $\emptyset$

The symbols in part (c) represent the *empty set* or *null set*.

b. Translating verbal phrases

Although this concept was introduced in *Mathematics 7 Syllabus*, the students need practice in translating verbal phrases into algebraic expressions. Phrases should include basic terminology related to operations, as well as the identification of unknown quantities which are written prior to solving verbal problems. Some students may find it necessary to think of similar but related problems involving numbers to recognize the arithmetic processes before they can write the expressions in algebraic form.

After an expression is formed, questions may be asked which vary from a simple evaluation of the phrase to others of a more challenging nature.

Example: Represent the sum of x and 3, the quantity divided by y .

Answer: $\frac{x+3}{y}$

Example: Represent the sum of x and the quotient of 3 and y .

Answer: $x + \frac{3}{y}$

Example: Write an expression to show the sum of the squares of two numbers.

Answer: $a^2 + b^2$

Example: Represent the number of kilograms in g grams.

Answer: $\frac{g}{1000}$, or $0.001g$

(How many kilograms are there in 2000 grams? in 3500 grams? in 400 grams?)

Example: Represent the number of days in W weeks and D days.

Answer: $7W + D$

Example: Bill earns 20 dollars more each week than Carl. If Bill earns d dollars each week, what is Carl's weekly salary?

Answer: $d - 20$

(If Bill earns \$160 each week, what is Carl's weekly salary?)

Example: All items at a clothing sale are listed at " $\frac{1}{3}$ off." If a jacket originally cost x dollars, what is its sale price?

Answer: $\frac{2}{3}x$

(If a jacket on sale costs \$120, what is its original price?)

The translation of verbal phrases to open sentences is discussed in this unit in two places: the writing of formulas and the solution of first-degree equations and inequalities. Certain language, however, tends to pose problems for the students, and it is suggested that the following translations be discussed at this stage. Ask the students which are phrases and which are sentences.

Verbal Phrase/Sentence	Algebraic Form
x decreased by 5	$x - 5$
x less than 5	$5 - x$
x is less than 5	$x < 5$
4 greater than y	$y + 4$
4 is greater than y	$4 > y$

2. Evaluating Algebraic Expressions

The order of operations studied in grade 7 is reviewed by substituting numbers for variables in algebraic expressions. The replacement method for solving equations and inequalities, discussed in *Mathematics 7 Syllabus*, may serve as another application of evaluating expressions. Many variations are possible in questions related to this topic.

Example: Evaluate $\frac{ab - bc}{c - a}$ when $a = -8$, $b = 5$, and $c = 7$

Answer: -5

Example: Given the domain $\{1, 2, 3, 4, 5, 6, 7\}$, solve for x : $\frac{2}{3}x - 1 < 2$

Answer: $\{1, 2, 3, 4\}$

Example: If $x = \frac{1}{2}$ and $y = 2$, which expression has the greatest value?

(a) xy (b) xy^2 (c) $(xy)^2$ (d) $-xy^2$

Answer: (b)

Example: For which values of a and b will $\frac{ab^2}{2}$ equal a whole number?

(a) $a = 3, b = 1$ (b) $a = 3, b = 2$ (c) $a = 1, b = 3$
(d) $a = 3, b = 3$

Answer: (b)

3. Translating Verbal Sentences into Formulas

In mathematics, a *formula* is an equation that expresses a relation between two or more variables. Each formula is a rule for operating with certain given variables. The students should know how to translate verbal sentences into formulas and how to express a formula in terms of words. Examples should include not only formulas from geometry but also relations using tables of measurement and everyday applications such as taxes, discounts, telephone charges, cab rates, and the like.

Example: Write a formula to show that the perimeter P of a rectangle is equal to the sum of twice its length L and twice its width W .

Answer: $P = 2L + 2W$

Example: Let m represent number of minutes, and h represent number of hours. Using m and h , write a formula to show the relation between minutes and hours.

Answer: $m = 60h$, or $h = \frac{m}{60}$

Example: A certain library book may be borrowed without charge for 28 days. The fine is \$.10 per day for each day that the book is overdue.

Write a formula to determine the fine F for a book that is d days overdue.

Answer: $F = .10d$

B. SOLVING FIRST-DEGREE EQUATIONS AND INEQUALITIES IN ONE VARIABLE

1. Properties of Equality

The students will use five basic properties of equality in solving equations in one variable. The properties are listed here for the teacher and should be developed as outcomes of problem solving. (Each property may be demonstrated by a physical model using a balance scale.)

a. Substitution property of equality

A number or expression may be substituted for its equal in any expression. (For example, on a balance scale, replace a 10 gram weight with two weights of 5 grams each.)

b. Addition property of equality

If $a = b$, then $a + c = b + c$.

c. Subtraction property of equality

If $a = b$, then $a - c = b - c$.

d. Multiplication property of equality

If $a = b$, then $ac = bc$.

e. Division property of equality.

If $a = b$, then $\frac{a}{c} = \frac{b}{c}$ (where $c \neq 0$).

Optional: A cancellation law may be applied to each of the last four properties of equality. For example, the cancellation law of addition states:

If $a + c = b + c$, then $a = b$.

2. A Plan for Solving Problems Algebraically

a. General technique

The students should learn how to use the tools of algebra to solve problems in general; they should *not* learn the mechanical processes often associated with "standard-type" problems. Review the general technique for problem solving as word problems are solved algebraically, and note specific tasks in the stages of planning and solving. The general plan of solution is presented at the left, as a problem is solved algebraically at the right.

GENERAL PLAN

Read: Read and understand the problem.

Plan: (a) Identify the variable, and express all unknowns in terms of this variable.

(b) Translate the relation in the problem from words to algebraic terms, forming an open sentence.

ALGEBRAIC SOLUTION

Example: Liz received 8 extra points for a report to bring her grade up to 92. What was her grade without the extra points?

Let x = grade without extra points. Then $x + 8$ = grade with points.

$$x + 8 = 92$$

Solve: (a) Apply the properties of equality (or inequality) to change the open sentence to its simplest equivalent form.

(b) Write the solution.

Check: Perform a check, to see if the solution makes sense.

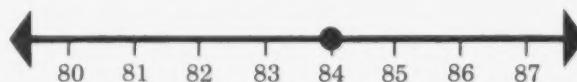
$$\begin{array}{rcl} x + 8 & = & 92 \\ x + 8 - 8 & = & 92 - 8 \\ x + 0 & = & 84 \\ x & = & 84 \end{array}$$

Solution: 84

$$\begin{array}{l} \text{Check: } 84 + 8 = 92 \\ \quad \quad 92 = 92 \text{ (True)} \end{array}$$

b. Equations

In the problem just presented, the open sentence $x + 8 = 92$ was transformed to the simpler equivalent sentences $x + 0 = 84$ and $x = 84$. *Equivalent sentences* are sentences that have the same solution. For this reason, $x + 8 = 92$ and $x = 84$ are called *equivalent equations*. Note that the subtraction property of equality, namely $x + 8 - 8 = 92 - 8$, could have been replaced by the addition property of equality, as in $x + 8 + (-8) = 92 + (-8)$. An alternative form of a final answer may be represented by a *number line graph*:



In solving word problems algebraically, limit the initial problems to those in which an equation reaches its simplest form in one step.

Example: In one day, the difference between the high and low temperatures was 7 degrees Celsius. Find the high temperature if the low temperature was 18°C .

Let h = high temperature

$$h - 18 = 7, \text{ or } h - 7 = 18$$

Use the addition property of equality.

Answer: 25°C

Example: In Miss Kandybowicz's Computer Club, $\frac{2}{3}$ of the students are girls. If there are 18 girls in the club, how many students are in the Computer Club?

Let t = total number of students in the club

Then $\frac{2}{3}t$ = number of girls in the club

$$\frac{2}{3}t = 18$$

Use the multiplication property of equality:

$$\frac{3}{2} \cdot \frac{2}{3}t = \frac{3}{2} \cdot 18$$

Answer: 27 students

c. Inequalities

Word problems should be extended to include inequalities that can be simplified in one step. Each *inequality* is a sentence that uses a symbol (usually $>$, $<$, $\geq$, or $\leq$) to state that two expressions are not equal.

Discuss the following properties of inequalities, related to the four basic operations, *ONLY* as they arise in problem-solving situations:

Addition properties of inequality

If $a > b$, then $a + c > b + c$.

If $a < b$, then $a + c < b + c$.

Subtraction properties of inequality

If $a > b$, then $a - c > b - c$.

If $a < b$, then $a - c < b - c$.

Multiplication properties of inequality

If $a > b$ and c is positive, then $ac > bc$.

If $a < b$ and c is positive, then $ac < bc$.

If $a > b$ and c is negative, then $ac < bc$.

If $a < b$ and c is negative, then $ac > bc$.

Division properties of inequality

If $a > b$ and c is positive, then $\frac{a}{c} > \frac{b}{c}$.

If $a < b$ and c is positive, then $\frac{a}{c} < \frac{b}{c}$.

If $a > b$ and c is negative, then $\frac{a}{c} < \frac{b}{c}$.

If $a < b$ and c is negative, then $\frac{a}{c} > \frac{b}{c}$.

The students should recognize the proper domain to use in each situation. Are the solutions only counting numbers, or positive numbers, or may there be any real numbers? Encourage number line graphs as alternative means of presenting solutions.

Example: During this season, the Wildcats won 5 more games than they lost. If the Wildcats won less than 9 games, how many games did they lose?

Let n = number of games lost

Then $n + 5$ = number of games won

"The number of games won is less than 9."

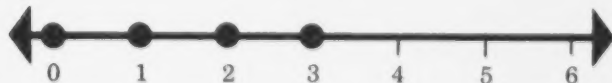
$$n + 5 < 9$$

$$n + 5 - 5 < 9 - 5$$

$$n < 4$$

Answer: 0, 1, 2, or 3 games lost

or,



Example: The length of a rug is 1.5 times its width. If the length is 6 meters or less, what is the width?

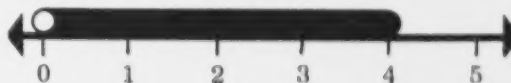
Let w = width of rug in meters

Then $1.5w$ = length of rug in meters

"The length is equal to or less than 6 meters."

$$\begin{array}{rcl} 1.5w & \leq & 6 \\ 1.5w & \leq & 6.0 \\ \hline 1.5 & & 1.5 \\ \hline w & \leq & 4 \end{array}$$

Answer: The width is less than or equal to 4 meters but greater than 0 meters, or



Not every equation or inequality must result from a word problem. Give the students practice in simplifying and solving open sentences of a general nature.

Unless otherwise stated, assume the domain to be the set of real numbers.

Example: Solve and check the following.

(a) $12 = y + 15$ (b) $\frac{3}{4}k < 24$

(c) $0.8m \geq 4$ (d) $-3x = -2$

Answer: (a) -3 (b) $k < 32$ (c) $m \geq 5$ (d) $\frac{2}{3}$

Example: If $8 + b = 3$, what is the value of $2b + 1$?

Answer: -9

3. Solutions of Open Sentences Involving Two or More Steps

a. Like terms

Two or more terms that contain the same variables and the same exponents for these variables are called *like terms*. The distributive property enables us to combine like terms under addition and subtraction.

$$10x + 4x = (10 + 4)x = 14x$$

$$7y - 9y = (7 - 9)y = -2y$$

$$8k + k = 8k + 1k = (8 + 1)k = 9k$$

Terms such as $4a$ and $4b$ are examples of *unlike terms* because the terms have different variables. The sum of $4a$ and $4b$ can only be expressed as $4a + 4b$; the terms cannot be combined further.

To simplify an equation containing like terms, first combine like terms and then apply inverse operations. These procedures should be studied both in problem-solving situations and with given equations.

Example: Together, a soda and a sandwich cost \$3.20. Find the price of each item if the sandwich costs 3 times as much as the soda.

$$\begin{aligned}\text{Let } c &= \text{cost of one soda} \\ \text{Then } 3c &= \text{cost of one sandwich} \\ c + 3c &= 3.20 \\ 4c &= 3.20 \\ c &= .80\end{aligned}$$

Answer: A soda costs \$.80 and a sandwich costs \$2.40.

Example: Solve for m and check: $2m + 27 = 5m$

$$\begin{aligned}2m + 27 &= 5m \\ -2m + 2m + 27 &= -2m + 5m \quad \text{or} \\ 27 &= 3m \\ 27 \div 3 &= 3m \div 3 \\ 9 &= m \\ \text{Check: } 2m + 27 &= 5m \\ 2(9) + 27 &= 5(9) \\ 18 + 27 &= 45 \\ 45 &= 45 \\ 2m + 27 &= 5m \\ \begin{array}{r} -2m \quad -2m \\ \hline 27 = 3m \\ 3 = 3 \end{array} \\ 9 &= m\end{aligned}$$

Answer: $m = 9$, or solution = $\{9\}$

b. Parentheses

An equation containing *parentheses* should first be transformed to an equation that does not contain parentheses.

Example: Mike, Fred, and Alice went bowling. Fred's score was 15 less than Mike's score. Alice's score was 3 times Fred's score. If Alice scored 165, what was Mike's score?

$$\begin{aligned}\text{Let } m &= \text{Mike's score} \\ \text{Then } m - 15 &= \text{Fred's score} \\ \text{Then } 3(m - 15) &= \text{Alice's score} \\ 3(m - 15) &= 165 \\ 3m - 45 &= 165 \\ 3m &= 210\end{aligned}$$

Answer: Mike scored 70.

Encourage the students to develop a general plan for simplifying open sentences, perhaps of the form:

- (1) Write the equation or inequality.
- (2) Use the distributive property to clear parentheses.

- (3) Combine like terms by addition or additive inverses.
- (4) Use the multiplicative inverse to isolate the variable.

Practice is needed in solving given equations and inequalities, as well as applying these skills to problem solving.

Example: A 747 airplane carries approximately 25,000 gallons of fuel. If this fuel weighs 167,500 pounds, find the weight of one gallon of fuel.

Answer: 6.7 pounds (Use $25,000p = 167,500$)

Example: It costs \$.40 for the first 3 minutes of a telephone call and \$.10 for each additional minute. How long can I talk for \$1.00?

Answer: Between 8 and 9 minutes, with a maximum of 9 minutes. (Use $.40 + .10(t - 3) \leq 1.00$)

Example: How many students are in the eighth grade if three-fifths of the grade is 30 more than two-fifths of the grade?

Answer: 150 students (Use $\frac{3}{5}g = 30 + \frac{2}{5}g$)

Example: Find the largest whole number x where $3(x + 2) < 30$.

Answer: 7

C. EQUATIONS AND THE COORDINATE PLANE

1. Functions of the Form $y = ax$, through Applications

Continue the pattern developed in grade 7 of introducing functions through applications.

Example: Joanna earns \$2.00 per hour for babysitting.

As the students consider this information (as well as other applications), they should perform the following specific activities:

- a. *Construct a table of values.*

Hours worked	0	1	2	3	4
Pay in dollars	0	2	4	6	8
- b. *Write in a descriptive sentence.*
 "The pay in dollars is twice the number of hours worked."
- c. *Develop a formula.*
 Select variables, and translate the sentence to algebraic form.

Let P = pay in dollars

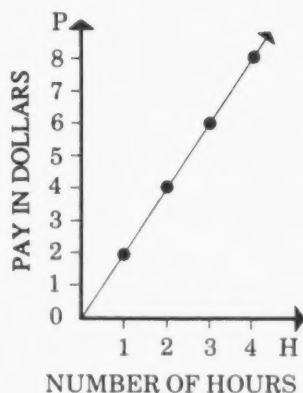
Let H = number of hours worked

$$P = 2H$$

- d. Draw a coordinate graph of the function.

The horizontal line or axis represents H , the vertical line or axis represents P , and points are plotted using ordered pairs or coordinates of the form (H, P) .

Since the points can be connected by a straight line, $P = 2H$ is an example of a linear equation or a linear function.



- e. Make predictions, based on the equation or the graph.

- (1) Given a value for H , find a corresponding value for P .

What pay should Joanna receive for $3\frac{1}{4}$ hours of work?

$$P = 2H$$

$$P = 2(3\frac{1}{4})$$

$$P = 6\frac{1}{2}$$

$$P = 6.50$$

Answer: \$6.50

- (2) Given a value for P , find a corresponding value for H .

If Joanna was paid \$11.00, how many hours did she babysit?

$$P = 2H$$

$$11 = 2H$$

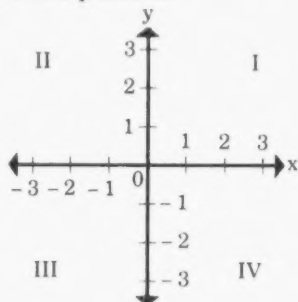
$$\frac{11}{2} = \frac{2H}{2}$$

$$5\frac{1}{2} = H$$

Answer: $5\frac{1}{2}$ hours

These concepts can be reinforced with other applications, as well as abstract tables and equations.

The students should know the four quadrants of the coordinate plane, and be able to plot points in all quadrants and on both the x -axis and the y -axis. All functions of the form $y = ax$ have graphs which pass through the origin $(0,0)$.



Application: There are 4 quarters in every dollar. Develop the function, $q = 4d$.

Application: When an object is dropped from a tall building on earth, its velocity or speed increases by 32 feet per second, for each second in motion. Develop the function, $v = 32t$.

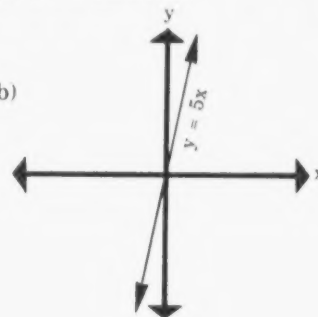
Time (t) seconds	Velocity (v) feet/sec.
0	0
1	32
2	64
3	96
4	128

Example: (a) After completing the table of values, write a formula to describe the function.

x	-3	-2	-1	0	1	2	3
y	-15			0	5	10	20

- (b) Draw a graph of the function.

Answer: (a) $y = 5x$ (b)



2. Functions of the Form $y = ax + b$, through Applications

The pattern of table, sentence, formula, graph, and predictions should be extended to linear functions whose graphs do not pass through the origin. Applications and abstract tables should be studied.

Application: It costs \$2.00 to rent a video cassette for 1 day. There is a charge of \$1.50 per day for each day that the cassette is kept beyond the 1st day. Using C = total cost and e = number of extra days, develop the function for quadrant I only:

$$c = 1.5e + 2$$

Application: At a theater, the cost for utilities, film rental, and salaries is \$180 per night. A ticket costs \$4. Using x = number of ticket buyers and y = profit/loss, develop the function:

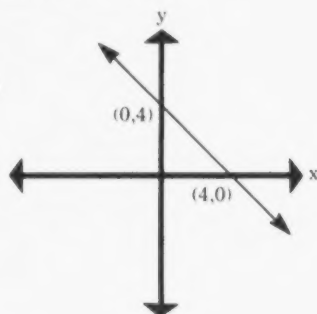
$$y = 4x - 180$$

(Why is the graph of this function acceptable in quadrants I and IV only? How many

tickets must be purchased to meet expenses? to make a profit?)

Example: (a) Write six ordered pairs (x, y) in which $x + y = 4$.
(b) Graph the function $x + y = 4$.

Answer: (a) varies (b)



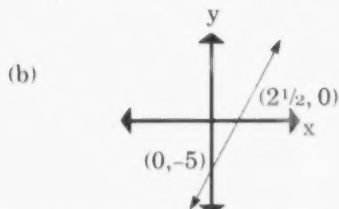
Example: Given the equation $y = 2x - 5$,
(a) Complete the table.

x	-1	0	1	2	3	4
y						

(b) Graph the function $y = 2x - 5$
(c) What is the value of y when $x = -8$?
(d) What is the value of x when $y = 9$?
(e) In which quadrants does the graph of $y = 2x - 5$ lie?

Answer: (a)

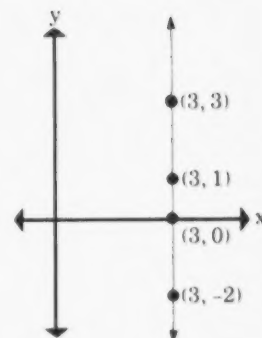
x	-1	0	1	2	3	4
y	-7	-5	-3	-1	1	3



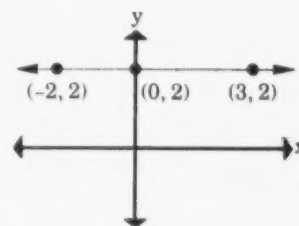
- (c) -21
(d) 7
(e) I, III, and IV

3. Coordinate Graphs of $x = a$ and $y = b$

At every point in the graph at the right, $x = 3$. In general, the equation of a line parallel to the y -axis (that is, a vertical line) is of the form $x = a$.



At every point in the graph at the right, $y = 2$. In general, the equation of a line parallel to the x -axis (that is, a horizontal line) is of the form $y = b$.

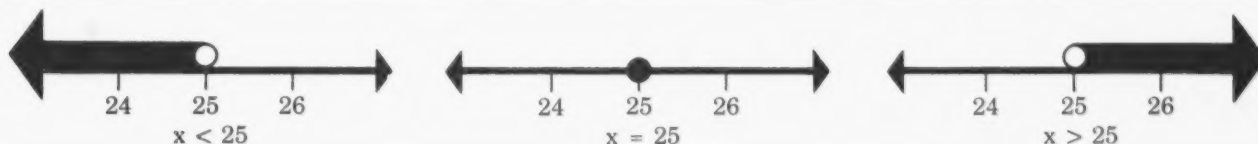


Questions concerning equation of these forms and their graphs should remain simple and direct.

D. INEQUALITIES AND THE COORDINATE PLANE

1. Number Line Graphs of Inequalities

On a number line, every darkened point or set of points indicates that the number or set of numbers is in the graph. Every nondarkened circle indicates that the number is not in the graph.

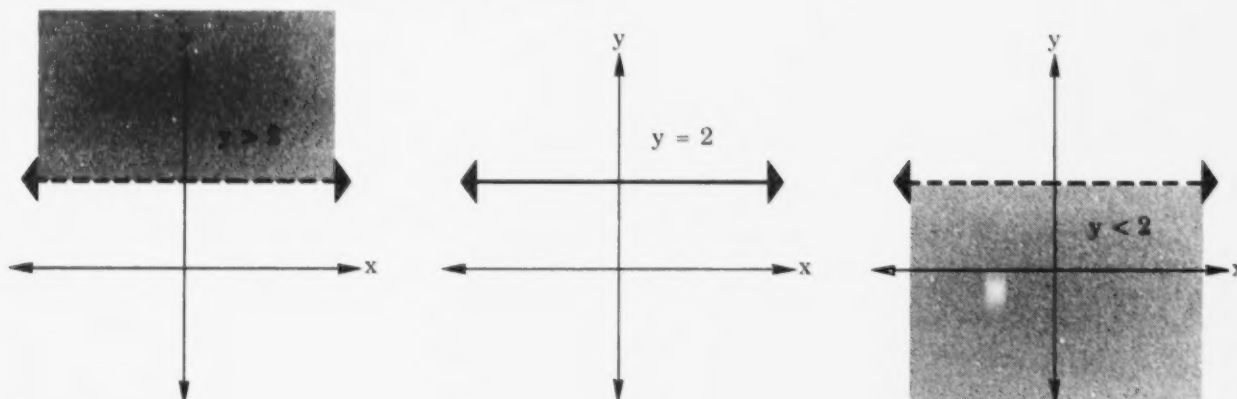


The union of the three graphs shown above is the entire number line. Other *compound relations* represent the union of two relations, and they are graphed below.

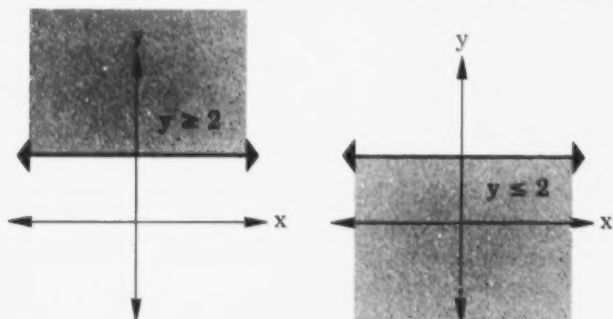


2. Optional: Coordinate Graphs of Inequalities

In the same way that a point separates a line into three parts (the point and 2 open rays), so too does a line separate a plane into three parts (the line itself and 2 half-planes).



Each half-plane is illustrated as a shaded region and represents an inequality. In a similar manner, as seen earlier, compound conditions can be graphed in the plane.

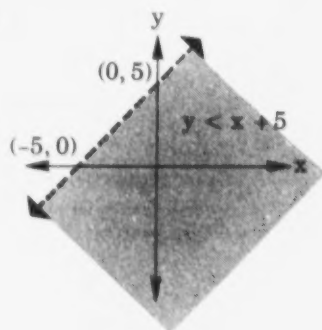


In effect, the graph of the line is the boundary for each half-plane. If any set of coordinates from a region satisfies the inequality, then the region is shaded as part of the graph.

In addition to the graphing of given inequalities, provide some practice that is application-oriented.

Example: Graph $y < x + 5$

Answer:



rate a plane into three parts (the line itself and 2 half-planes).

3. Applications of Inequalities

Application: The school cafeteria makes less than 2 sandwiches per student on a daily basis.

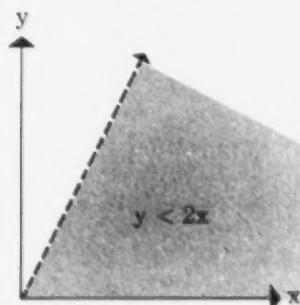
Let x = number of students

Let y = number of sandwiches

"Number of sandwiches is less than 2 times the number of students."

$$y < 2x$$

This graph is restricted to the first quadrant, because the number of students and number of sandwiches cannot be negative.



VI. COMPUTER AWARENESS

This unit is divided into three sections, the first of which contains some introductory remarks and an overview. In the second section, mathematical operations, functions, and relations are discussed from a computer viewpoint. Whether or not computers are available, the students must gain some knowledge of the operations and functions used in a programming language to allow them to study algorithms (plans for solving problems).

The third section of this unit, namely computer programming, is optional only because of equipment availability. In schools where computers are available, all students should have access to these devices.

It is recognized that some schools present this material in a separate curriculum outside of the regular mathematics classroom. In such a case, it is not necessary to repeat these sections.

Note: The Computer strand may be taught as a unit or woven throughout the curriculum at appropriate places. Commercially prepared programs can supplement the course content and increase the students' exposure to computers.

A. INTRODUCTION TO COMPUTERS

This section, including time devoted to classroom discussion and/or student reports, should be completed in 1-2 days.

1. Optional: Components of a Modern-Day Digital Computer

Review the five basic components of a digital computer, all of which have been defined in *Mathematics 7 Syllabus*:

- (1) Input
- (2) Control
- (3) Arithmetic and Logic Unit
- (4) Memory
- (5) Output

2. Optional: Careers

As the computer becomes more prevalent in our society, today's students will most likely be employed as adults in fields where the computer plays some role, either directly or indirectly. Jobs and titles that do not exist today will emerge in the next 10-15 years because of the computer. Rather than list specific titles and job descriptions, the teacher should describe career categories. The teacher should also discuss the fact that adaptive devices have al-

lowed many individuals with handicapping conditions to participate in fields which use computers.

Careers related directly to the computer fall into four categories:

- (1) Designing and building computers. (Hardware)
- (2) Writing programs for the computer. (Software)
- (3) Operating computer systems. (Note that a *computer system* consists of both equipment and programs, that is, hardware and software.)
- (4) Servicing and repairing computer systems.

Many more careers are influenced indirectly by the computer, as people use the machine and its programs to do specific tasks. Have some students report on ways in which their parents or neighbors have jobs that have changed somehow because of the computer: office management, small businesses, banks, hospitals, and the like. Have some students research the adaptive devices referred to above.

3. The Loop as a Computer Process

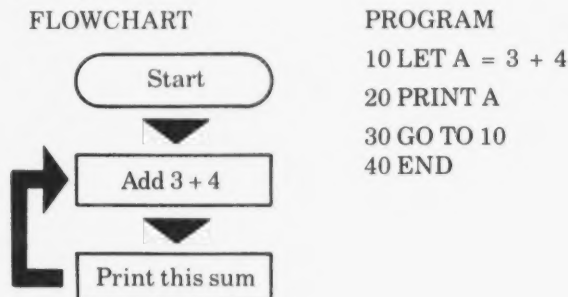
Although the computer is recognized as a tool for solving complex problems, a computer also handles problems of a repetitive nature.

a. Flowcharts

In grade 7, flowcharts with loops were introduced. In grade 8, the students may construct flowcharts for various activities, each involving a loop to show a repeated process. For example, describe how daily attendance is taken, or how lunch is sold in the school cafeteria.

The teacher should present at least two examples of loops in a mathematical setting: in one case, an endless loop, and in another case, a loop conditioned to stop at some point. These examples can be studied through the use of flowcharts, or programs, or both.

Example:



Output:

```

7
7
7
7
7

```

Follow the flowchart or, as explained here, the BASIC program. The computer determines on line 10 that $A = 7$ and, following the directions on line 20, it prints 7. When line 30 is reached, the computer "loops back" to line 10 and repeats the process. Here, 7 is printed endlessly and line 40 is never reached. Depending upon the machine being used, a certain key is hit to STOP the output and ESCAPE the loop.

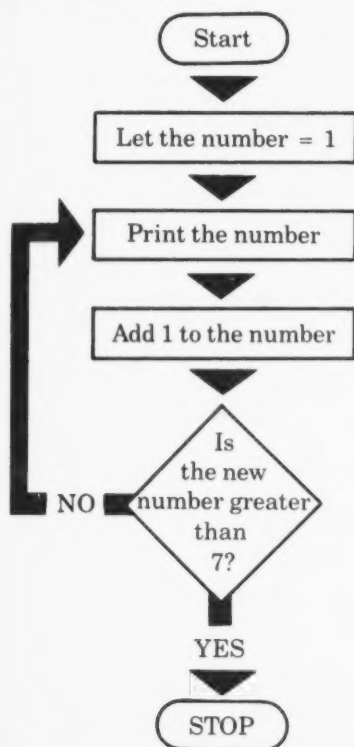
til all integers from 1 through 7 are printed. At that point, the computer moves to line 40 and the END of the program.

b. Optional: Programs

If computers are available, the students should program a number of variations on this theme. It is recommended, however, that statements of the form $N = N + 1$ not be discussed at this level to avoid confusion that may occur as the students begin to learn the foundations of algebra.

The students may be shown programs containing either finite or endless loops and asked to alter them. The students should also discover how spacing in a display can be varied by using commas and semicolons. Programs containing endless loops, for example, can result in interesting geometric patterns found in the displays of their outputs. Depending upon the computer being used, how do the outputs of the following programs differ?

10 PRINT "MATH"	10 PRINT "MATH ",	10 PRINT "MATH ";
20 PRINT "IS"	20 PRINT "IS ",	20 PRINT "IS ";
30 PRINT "FUN"	30 PRINT "FUN ",	30 PRINT "FUN ";
40 GO TO 10	40 GO TO 10	40 GO TO 10

Example:**FLOWCHART****PROGRAM**

```

10 FOR N = 1 TO 7
20 PRINT N
30 NEXT N
40 END

```

B. MATHEMATICAL OPERATIONS, FUNCTIONS, AND RELATIONS

This section may be taught without the use of a computer and, excluding any time devoted to programming, should be completed in 6-7 days. **Optional:** If computers are available, however, they should be accessed by the students. Necessary adaptive devices may be used to ensure that students with handicapping conditions have equal access to school computers. Suggestions for programs, found in Part III of this unit, should be incorporated within Part II for students of average or higher ability. This programming should increase the time requirements by an additional 5 days. Students who might not be able to program at this level should still access the computer to perform appropriate activities such as using commercially prepared programs, CAI, teacher-constructed programs, or using the computer for computational purposes.

1. Operations Using Computer Symbols

Review the operational symbols used on the existing computer at the school, or present the following symbols from BASIC:

Addition:	$A + B$
Subtraction:	$A - B$
Multiplication	$A * B$

Output:

```

1
2
3
4
5
6
7

```

Follow the flowchart or, as explained here, the program. The FOR/NEXT loop in this BASIC program repeats a process exactly seven times. At the start, $N = 1$ and, according to line 20, 1 is printed. On line 30, the NEXT integer or integral value of N is 2. The computer "loops back" to step 10, prints 2 according to line 20, and takes on the next integral value for N , namely 3. The process repeats un-

Division	A/B
Raising to a power:	$A \wedge B$, or $A \uparrow B$, or $A^{**}B$ (depending upon computer)

2. Evaluating Expressions in Computer Format

This material may be integrated with other strands of the *Syllabus* or studied here as reinforcement of concepts presented earlier.

a. Order of operations

Expand the activities begun in grade 7 to include negative numbers. The students should realize that the order of operations is normally incorporated into the structure of the computer, and they should evaluate numerical expressions using operational symbols from a programming language.

Example: Evaluate each of the following expressions, which use symbols from the BASIC language:

- (a) $2 - 3 * 4$ (b) $-5 - 8$ (c) $-30/5$
 (d) $8 - 6/2 - 4$ (e) $(8 - 6)/(2 - 4)$ (f) $(-2) * (-3)$

Answer: (a) -10 (b) -13 (c) -6
 (d) 1 (e) -1 (f) .06

Optional: If a computer is available, the students should check their responses by using a simple PRINT statement, and they should know how to translate standard operational symbols into the symbols of a programming language.

Example:

- (a) Which of the following expressions has the smallest value?
 (1) -2^3 (2) -3^2 (3) $(-2)^3$ (4) $(-3)^2$
 (b) Check your response by evaluating the expressions on a computer.

Answer: (a) (2)
 (b) (1) PRINT $-2 \uparrow 3$
 - 8
 (2) PRINT $-3 \uparrow 2$
 - 9
 (3) PRINT $(-2) \uparrow 3$
 - 8
 (4) PRINT $(-3) \uparrow 2$
 9

b. Use of decimals in output

Remind the students that the computer will output any rational number which is not an integer in decimal nota-

tion. Rounding errors can also occur in outputs since computers can display only a limited number of digits.

Example: On Monday, a stock sold at $12\frac{3}{8}$ per share. On Friday, the stock sold at $11\frac{7}{8}$ per share.

- (a) Which of the following computer expressions correctly calculates the change in the worth of the stock from Monday to Friday?

- (1) PRINT $12 + \frac{3}{8} - 11 + \frac{7}{8}$
 (2) PRINT $(12 + \frac{3}{8}) - (11 + \frac{7}{8})$
 (3) PRINT $11 + \frac{7}{8} - 12 + \frac{3}{8}$
 (4) PRINT $(11 + \frac{7}{8}) - (12 + \frac{3}{8})$

- (b) How would the computer display this change?

Answer: (a) (4)
 (b) -.5, or -0.5

Example: The following unit fractions will be displayed by a computer in decimal form: $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, $\frac{1}{9}$, $\frac{1}{10}$. Imagine that the computer display is *restricted to eight digits*.

- (a) How many of the fractions will be shown in their correct decimal equivalent form? Name these fractions and their decimal equivalents.
 (b) Which fraction will be displayed as .16666667?

Answer: (a) Five fractions: $\frac{1}{2} = .5$ $\frac{1}{4} = .25$
 $\frac{1}{5} = .2$ $\frac{1}{8} = .125$
 $\frac{1}{10} = .1$

- (b) $\frac{1}{6}$ (Discuss the rounding error.)

Optional: If a computer is available, a simple but effective program including a FOR/NEXT loop can be used to study the example just cited.

```
10 FOR N = 2 TO 10
20 PRINT 1/N
30 NEXT N
40 END
```

Output can be enhanced by improving the PRINT statement:

```
20 PRINT "1/" ; N ; " = " ; 1/N
```

c. Use of scientific notation in output

The exponential form, or E-form, was introduced in grade 7 to show how a computer displays large numbers, specifically values greater than or equal to 1. Now that scientific notation has been studied in grade 8 to include very small values (that is, values between 0 and 1), similar problems should be studied using E-notation. Note that E represents a base of 10 and that the numeral following E, whether negative or positive, indicates the exponent.

Example: Given the number .0000000135,

- Write this number in scientific notation.
- Write this number in the E-form used by a computer.

Answer: (a) 1.35×10^{-8}

(b) $1.35 \text{ E} - 8$

Example: If a computer display is restricted to eight digits, find the output when E notation is used:

- PRINT .004 + 3
- PRINT .0000072/100000

Answer: (a) $6.4\text{E} - 8$

(b) $7.2\text{E} - 11$

3. Functions Used with Computers

Present each function as it is entered on the computer at school or, if no computer is available, use the BASIC form of the function:

a. Absolute value function

Review the absolute value function: $\text{ABS}(X)$

Present situations in which the concept of an absolute value is used.

Example: During a scientific experiment, the height of water in a beaker was measured at two stages. If these readings were 3.58 cm and 2.9 cm, what was the difference of these measures?

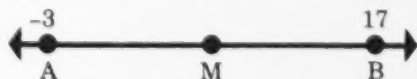
Answer: It was not noted whether water was added or whether it evaporated. By the absolute value function,

$$\text{ABS}(3.58 - 2.9) = \text{ABS}(.68) = .68$$

$$\text{ABS}(2.9 - 3.58) = \text{ABS}(-.68) = .68$$

In either case, the difference is .68 cm.

Example: On a number line, point A corresponds to -3, and point B corresponds to 17.



- Find the average of -3 and 17, and assign this value to point M on the number line.
- Demonstrate that M is the midpoint of AB by showing that the distance from A to M equals the distance from M to B. Hint: Use absolute value.

Answer: (a) $(-3 + 17)/2 = 14/2 = 7$. Assign 7 to point M.

$$\begin{aligned} \text{(b) } \text{ABS}(A - M) &= \text{ABS}(-3 - 7) = \\ &\text{ABS}(-10) = 10 \end{aligned}$$

$$\begin{aligned} \text{ABS}(M - B) &= \text{ABS}(7 - 17) = \\ &\text{ABS}(-10) = 10 \end{aligned}$$

b. Greatest integer function

This function was first introduced in grade 7. The algorithms which were studied to round positive numbers to the nearest integer or to the nearest ten, and to test for divisibility, were listed as optional topics.

Review the definition and standard BASIC notation for the greatest integer function:

$\text{INT}(X)$

The greatest integer of X is *either* X itself, if X is an integer, *or* the greatest integer less than X . The domain of X , which was restricted to positive numbers in grade 7, should now be expanded to include negative numbers. In evaluating the function for non-integer values of X , some students may benefit from such visual aids as a number line (move to the left) or a thermometer (move down).

$$\text{INT}(19) = 19 \quad \text{INT}(-5) = -5 \quad \text{INT}(2.9) = 2$$

$$\text{INT}(2/3) = 0 \quad \text{INT}(-.4) = -1 \quad \text{INT}(-3 \frac{1}{3}) = -4$$

Algorithm: Rounding to the nearest integer

In grade 7, for any positive value of X an algorithm was given to round X to the nearest integer:

$$\text{INT}(X + .5)$$

After the students review the algorithm for positive values of X , ask them if this rule will hold true for negative values of X . The answer is "Yes," as illustrated below:

$$\begin{aligned} \text{If } X = -3.1, \text{ then } \text{INT}(X + .5) &= \text{INT}(-3.1 + .5) = \\ &\text{INT}(-2.6) = -3 \end{aligned}$$

$$\begin{aligned} \text{If } X = -3.7, \text{ then } \text{INT}(X + .5) &= \text{INT}(-3.7 + .5) = \\ &\text{INT}(-3.2) = -4 \end{aligned}$$

To the nearest integer, -3.1 is rounded to -3, and -3.7 is rounded to -4.

Optional: Other rounding algorithms can be studied and tested for negative as well as positive values of X . Work from innermost parentheses first, and perform one computation at a time; note that similar steps are taken by the computer in evaluating these expressions. For example:

Algorithm: Rounding to the nearest ten

$$\text{INT}(X/10 + .5)*10$$

or

$$\text{INT}((X + 5)/10)*10$$

Algorithm: Rounding to the nearest tenth

INT (10*X + .5)/10

or

INT (10*(X + .05))/10

c. Square root function

The square root function, which lists the principal square root of a number, is usually indicated in BASIC by the symbols:

SQR (X)

The principal square root of X is one of two equal non-negative factors whose product is X. Discuss the principal square root as the inverse of squaring a number that is positive or 0. Provide practice in evaluating expressions.

In standard notation: $7^2 = 49$, and $\sqrt{49} = 7$

In BASIC notation: $7 \uparrow 2 = 49$, and $\text{SQR}(49) = 7$

Example: Evaluate the following expressions:

- (a) SQR (144) (b) SQR (0) (c) SQR (900)
(d) SQR (.36) (e) SQR (6.25) (f) SQR (5 $\uparrow$ 2)

Answer: (a) 12 (b) 0 (c) 30
(d) .6 (e) 2.5 (f) 5

Geometric formulas, including some for area and the Pythagorean Theorem, provide opportunities to use the square root function.

Example: The area of a square is 14.44 cm^2 .

- (a) Write a computer statement to find the length of one side of the square.
(b) What is the length of one side of the square?

Answer: (a) PRINT SQR (14.44)
(b) 3.8 cm

The computer will display a rational approximation for an irrational number. Consider irrational numbers, such as $\sqrt{2}$ and $\sqrt{3}$, and the statements needed for computer input:

PRINT SQR (2)
PRINT SQR (3)

The number of digits displayed as output will vary from one computer to another.

d. Random number function

Be aware that the random number generator varies from one computer to another. In some cases, RND or RND (0) or RND (1) will generate a random number greater than 0 and less than 1. On some machines, if used

again, a new random number will appear; but on other computers, the same number that appeared the first time will be generated again. To avoid this situation, input the word RANDOM or RANDOMIZE so that the set of random numbers is "shaken up" and a new number appears.

If computers are available in the school, teach the random number function for the existing system. If no computers are available, use the following definition which will be assumed for the examples in this *Syllabus*:

RND (1)

RND (1) will generate a random number greater than 0 and less than 1. Each time RND (1) is used, a new random number, unrelated to any before it, will be generated.

Ask the students how five random numbers could be displayed without printing RND (1) five times in a row. Study the program and its output. Here it is assumed that the computer displays an 8-digit numeral.

Program:

```
10 FOR N = 1 TO 5
20 PRINT RND (1)
30 NEXT N
40 END
```

Output:

.73856122
.93104276
.11897505
.47428564
8.2345617E-2

(Note that the last random number is actually .082345617.)

Consider how the output would have been altered if the PRINT statement on line 20 were changed. Ask the students to give the output to each program, using the set of five random numbers generated above.

Program:

```
10 FOR N = 1 TO 5
20 PRINT 10*RND (1)
30 NEXT N
40 END
```

Output:

7.3856122
9.3104276
1.1897505
4.7428564
0.8234562

In the program above, the computer has now generated five random numbers greater than 0 and less than 10.

Program:

```
10 FOR N = 1 TO 5
20 PRINT INT (10*RND (1))
30 NEXT N
40 END
```

Output:

```
7
9
1
4
0
```

By using the greatest integer function in the program above, the computer has generated five integers at random. The smallest integer is 0, and the largest possible integer is 9.

Program:

```
10 FOR N = 1 TO 5
20 PRINT INT (10*RND(1)) + 1
30 NEXT N
40 END
```

Output:

```
8
10
2
5
1
```

In the program just written, each integer was increased by 1. The smallest possible integer in the output will be 1, while the largest possible integer is 10. In other words, the program generates the integers or whole numbers from 1 to 10. This leads to the following algorithm:

Algorithm: To select a number from 1 through 10 at random.

$\text{INT}(10 \cdot \text{RND}(1)) + 1$

Similar algorithms can be written and used in programs to simulate experiments in probability, such as tossing a die (1, 2, 3, 4, 5, 6) and flipping a coin (0, 1 for head, tail). See section C of this unit for sample programs using these algorithms.

Algorithm: To select a number from 1 through 6 at random.

$\text{INT}(6 \cdot \text{RND}(1)) + 1$

Algorithm: To randomly select 0 or 1.

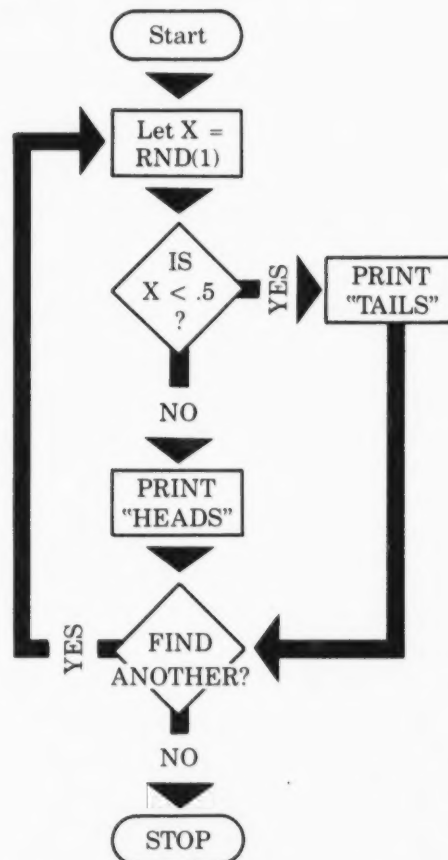
$\text{INT}(2 \cdot \text{RND}(1))$

4. Optional: Relations Used with Computers

a. Simple inequalities

Inequalities such as *greater than* and *less than* are used in a number of computer applications. These inequalities are usually related to decision boxes in a flowchart.

The flowchart simulates an experiment in which a coin is tossed. If $\text{RND}(1)$ is less than .5, an output of "tails" is to be printed and if $\text{RND}(1)$ is greater than or equal to .5, an output of "heads" is to be printed.



If computers are available, a program for this flowchart may be presented so that the students can analyze it and run it. The following program assumes 100 tosses of the coin.

```
10 FOR N = 1 TO 100
20 LET X = RND(1)
30 IF X < .5 THEN 60
40 PRINT "HEADS"
```

```

50 GO TO 70
60 PRINT "TAILS"
70 NEXT N
80 END

```

b. Compound inequalities

Some students may wish to work with compound inequalities in a program. They should be shown the standard form in BASIC or the symbolism for the local computer system in the school.

Algebraic Form

$x \geq 3$
 $y \leq 5$
 $k \neq 12$

BASIC Form

$X > = 3$
 $Y < = 5$
 $K < > 12$

C. Optional: COMPUTER PROGRAMMING

If computers are available in the school, the students should have hands-on experience with these devices. This section on programming is best treated if it is integrated with Section II of this unit.

1. Determine Computer Output of Given Programs

Before doing self-programming, the students should have some understanding of programs presented by the teacher.

Example: List the output without actually running the program.

Program A

```

10 LET X = SQR (INT (16.73))
20 LET Y = INT (SQR (31))
30 PRINT X
40 PRINT Y
50 PRINT ABS (X - Y)
60 END

```

Output

```

4
5
1

```

Program B

```

10 FOR N = 1 TO SQR (16)
20 PRINT N/2
30 NEXT N
40 END

```

Output

```

.5
1
1.5
2

```

2. Write Original Programs

Many students should be able to write programs which use the functions studied in section B of this unit. In some situations, the teacher may write programs and have the students enter and run them.

Example: Write a program to print a table of the integers from 1 through 50, and the square root of each integer.

Program:

```

10 PRINT "N," "SQR(N)"
20 FOR N = 1 TO 50
30 PRINT N, SQR(N)
40 NEXT N
50 END

```

Example: Write a program to simulate the toss of a die 40 times.

Program:

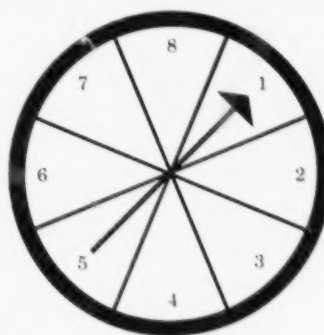
```

10 FOR N = 1 TO 40
20 PRINT INT (6*RND(1)) + 1
30 NEXT N
40 END

```

The following list contains just a few additional program activities which may be considered for 8th-grade students. For each, write a program to do the following:

- Evaluate perimeter and area formulas for given triangles and quadrilaterals.
- Round any number to the nearest hundred or thousand (or to the nearest hundredth, or thousandth).
- Given the lengths of any two sides of a right triangle, find the length of the third side.
- Use the compound interest formula to determine various amounts of money earned under given conditions.
- The spinner contains 8 regions, each equally likely to occur when the arrow is spun. Simulate 60 spins of the arrow and record the outcomes randomly generated.



- Generate 20 random numbers which are between 0 and 13.
- Generate 20 random numbers which are integers between 0 and 13.
- Let the computer randomly generate a whole number less than 100, but not display it. The user can

guess at the number, and the computer will respond "TOO HIGH" or "TOO LOW" for each guess taken until the number is guessed.

- Given an input of any three numbers, have the computer arrange them in order from lowest to highest.

VII. GEOMETRY AND MEASUREMENT

In grade 7, the students began to understand and use proper terminology, symbols, and formulas in geometry. The study of geometry should continue to offer avenues for exploratory activities and experiments as its foundation is formally developed. Diagrams, patterns, and geometric properties are tools often used in problem solving.

A number of concepts in grade 8 are extensions of ideas developed in grade 7 and, in this light, there is a need to review basic definitions.

If the systems of measure found in section B of this unit have been studied earlier, the teacher may wish to integrate the measurement of geometric figures from section C with the development of geometry in section A. Constructions serve as another basis for the integration of topics in geometry.

A. GEOMETRY

1. Points, Lines, and Planes

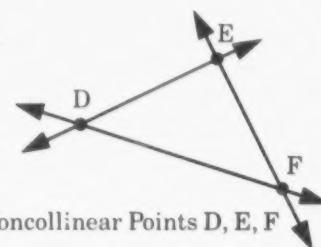
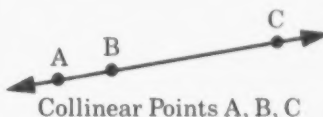
Recall the three *undefined terms* of geometry: *point*, *line* (which is understood to mean straight line), and *plane*. See *Mathematics 7 Syllabus* for a discussion of these terms.

a. Points: collinear, noncollinear, coplanar

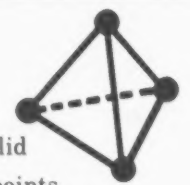
Collinear points are those points contained in the same straight line. For any two points, there is one and only one

straight line that contains both points. In other words, two points determine a line. However, since a line is an infinite set of points, each line contains more than two points.

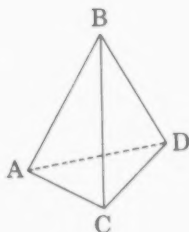
Noncollinear points are three or more points that are not contained in the same line. Three noncollinear points determine a plane.



Coplanar points are those points contained in the same plane. Three noncollinear points, as shown in the diagram at the upper right, must be coplanar. While four or more noncollinear points may lie in the same plane, select a fourth point that is not in the plane of the first three. These four points describe a 3-dimensional figure called a geometric solid. Consider the following relations:

 Point 1 point 0 dimensions	 Line 2 points 1 dimension	 Plane 3 points 2 dimensions	 Solid 4 points 3 dimensions
---	--	---	--

Example: The diagram represents a 3-dimensional object. Using the letters given in the diagram, name all possible points, lines, and planes.



Answer: 4 points: A, B, C, D

6 lines: $\overline{AB}$, $\overline{AC}$, $\overline{AD}$, $\overline{BC}$, $\overline{BD}$, $\overline{CD}$

4 planes: plane ABC, plane ABD, plane BCD, plane ACD

b. Lines, segments, and rays

The students should know that a *segment* is a finite portion of a line and has two endpoints. A *ray* is an infinite portion of a line and has only one endpoint. Appropriate symbolism should be used in identifying lines, segments, and rays.

Example: In the diagram, points A, B, C, D, E, F, G, and H are collinear.



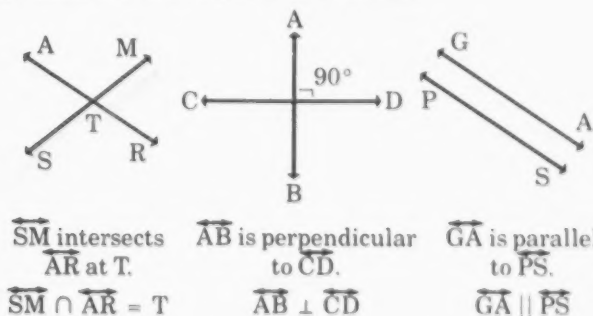
Using only these points, tell: (a) how many different rays can be identified and, (b) how many different segments can be named.

Strategy: Use simpler but related problems, leading to the recognition of certain sets of numbers.

Collinear Points		Rays	Segments
$\overleftrightarrow{AB}$	2	2: $\overleftrightarrow{AB}, \overleftrightarrow{BA}$	1: $\overline{AB}$
$\overleftrightarrow{ABC}$	3	4: $\overleftrightarrow{AC}, \overleftrightarrow{BC}, \overleftrightarrow{CA}, \overleftrightarrow{BA}$	3: $\overline{AB}, \overline{BC}, \overline{AC}$
$\overleftrightarrow{ABCD}$	4	6: $\overleftrightarrow{AD}, \overleftrightarrow{BD}, \overleftrightarrow{CD}, \overleftrightarrow{DA}, \overleftrightarrow{CA}, \overleftrightarrow{BA}$	6: $\overline{AB}, \overline{BC}, \overline{CD}, \overline{AC}, \overline{BD}, \overline{AD}$
	5	8	10
	6	10	15
	7	12	21
	8	14	28
	(Even Numbers)		(Triangular Numbers)

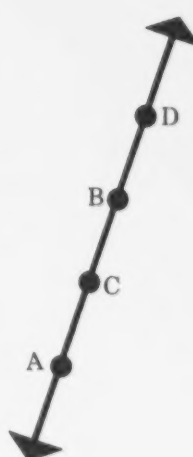
c. Pairs of lines: intersecting, perpendicular, parallel, skew

In grade 7, the students considered pairs of lines in a plane. Two lines are called *intersecting* if and only if they meet at one point. Two lines are *perpendicular* if and only if they intersect to form right angles. Two lines in the same plane that never meet are called *parallel* lines. The symbols for these relations, previously treated as being optional, are now required to be known.



Optional:

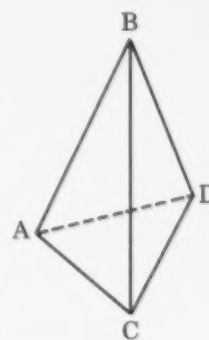
If two lines coincide or occupy the same place, they cannot be called intersecting because they meet at more than one point. Consider $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$



as shown at the left. At this level, the students may be content in saying that " $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ coincide," or that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ name the same line." The teacher may choose to introduce the concept that *every line is parallel to itself*. In other words, " $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$." This leads to a definition of parallel lines in higher grades wherein parallelism is an equivalence relation (reflexive, symmetric, and transitive): Two lines are *parallel* if and only if the lines lie in the same plane and do not meet or the lines coincide.

The students should know the meaning of *horizontal* and *vertical* lines. A line which is neither horizontal nor vertical is said to be *oblique*. A pair of *oblique* lines may be described as two lines that meet neither at a right angle nor at a straight angle.

Coplanar lines are those lines contained in the same plane. In the accompanying diagram of a solid, $\overleftrightarrow{BC}$, $\overleftrightarrow{CD}$, and $\overleftrightarrow{BD}$ are coplanar lines because they lie in plane BCD.

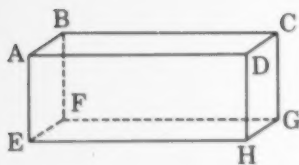


Skew lines are lines that lie in different planes so that the lines do not intersect and are not parallel. In the solid, $\overleftrightarrow{BC}$ and $\overleftrightarrow{AD}$ are skew lines. Also, $\overleftrightarrow{BD}$ and $\overleftrightarrow{AC}$ are skew, and $\overleftrightarrow{BA}$ and $\overleftrightarrow{CD}$ are skew.

The intersection of two planes in space is a line. For example, in the solid shown above, the intersection of plane ABC and plane ADC is $\overleftrightarrow{AC}$. Two planes and the intersecting line can be represented by simply folding a piece of paper.

Two planes in space that never meet are called *parallel planes*. Consider the planes that describe the ceiling and floor of a typical room. In fact, a rectangular prism, whether it be the classroom itself or a shoe box, can serve as a model from which many relationships of lines and planes in 3-dimensional space can be discussed and illustrated.

Example: Points A, B, C, D, E, F, G, and H identify the vertices or corners in our classroom. Use these points to identify points, lines, and planes in answering questions.



Answer the question, or fill in the blank.

Answer:

- | | |
|---|---|
| (a) Points D, C, G, and ____ are coplanar. | (a) H |
| (b) Points B, C, H, and ____ are coplanar. | (b) E |
| (c) Which lines are parallel to $\overline{EH}$? | (c) $\overline{FG}$, $\overline{AD}$, $\overline{BC}$ |
| (d) $\overline{EH}$ and $\overline{CG}$ are _____ lines. | (d) skew |
| (e) Which lines are coplanar with $\overline{AB}$ and $\overline{BF}$? | (e) $\overline{AE}$ and $\overline{EF}$
(Also, $\overline{AF}$ and $\overline{BE}$.) |
| (f) Which lines are coplanar with $\overline{AD}$ and $\overline{FG}$? | (f) $\overline{AF}$, $\overline{DG}$, $\overline{AG}$, $\overline{DF}$ |
| (g) Plane $ADHE \cup$ plane $DCGH =$ _____ | (g) $\overline{DH}$ |
| (h) Which plane is parallel to $BCGF$? | (h) plane $ADHE$ |
| (i) Which lines are perpendicular to $\overline{EH}$? | (i) $\overline{AE}$, $\overline{DH}$, $\overline{FE}$, $\overline{GH}$
(Also, $\overline{BE}$ and $\overline{CH}$ in $BCHE$.) |
| (j) Which lines are skew to $\overline{DH}$? | (j) $\overline{AB}$, $\overline{EF}$, $\overline{BC}$, $\overline{FG}$
(Also, $\overline{AC}$, $\overline{AF}$, $\overline{AG}$, $\overline{BE}$, $\overline{BG}$, $\overline{CE}$, $\overline{CF}$, $\overline{EG}$) |
| (k) Eight points are used to identify the rectangular prism. Using only these points, how many different segments can be named? | (k) 28, which is the same number of segments from eight collinear points seen earlier. |

2. Angles

An *angle* is the set of points contained in two rays that have a common endpoint.

a. Measure of an angle

In grade 7, the students were required to know the parts of an angle and the classification of angles according to their measures: *acute*, *right*, *obtuse*, and *straight*. In grade 8, the teacher should introduce the symbol for the *measure* of an angle and use it throughout this unit. If the "measure of angle ABC is 90 degrees," write:

$$m\angle ABC = 90$$

b. Angle pairs

It may be necessary to review definitions concerning pairs of angles:

Complementary angles are two angles whose sum is equal to a right angle.

Supplementary angles are two angles whose sum is equal to a straight angle.

Adjacent angles are two angles that share a common ray and a common vertex but have no interior points in common.

Vertical angles are nonadjacent pairs of angles formed by two lines that intersect.

These concepts should be applied to a series of problems in which an understanding of these terms is required to find solutions. Although the strategy used in every problem need not be algebraic in nature, algebraic solutions should be required in a number of cases.

Example: If $\overline{AB}$ and $\overline{CD}$ intersect at point E, and the measure of $\angle AED$ is 55° , find: (a) $m\angle DEB$ (b) $m\angle CEB$

Answer: (a) 125 (b) 55

Example: Two angles are complementary. If the measure of one angle is 20° more than the measure of the other, find the measures of the angles.

Answer: 35° and 55°

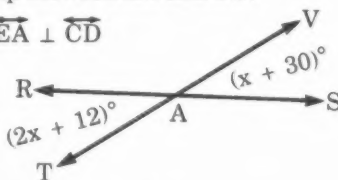
Example: Let $\angle CEA$ and $\angle DEA$ be adjacent supplementary angles that are equal in measure. (a) Find $m\angle CEA$. (b) Describe the relationship between $\overline{EA}$ and $\overline{CD}$.

Answer: (a) 90 (b) $\overline{EA} \perp \overline{CD}$

Example: In the diagram, $\overline{RS}$ intersects $\overline{TV}$ at A, $m\angle SAV = x + 30$, and $m\angle TAR = 2x + 12$.

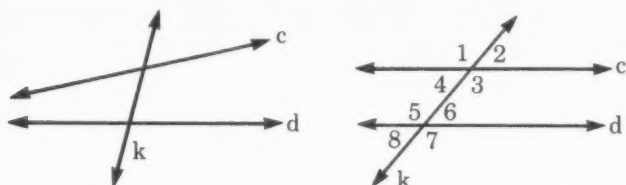
(a) Find x . (b) Find $m\angle TAR$. (c) Find $m\angle SAT$.

Answer: (a) 18 (b) 48 (c) 132



3. Parallel Lines Cut by a Transversal

A *transversal* is a line that intersects two other lines at two distinct points. In each diagram below, line k is a transversal to lines c and d . As an introductory step, the students may be asked to use protractors to measure the eight angles formed in each diagram: first where lines c and d are not parallel, and then where $c \parallel d$.



The numbered angles, shown in the right-hand diagram, should assist the students as they refer to these angles and their relations.

If and only if two parallel lines are cut by a transversal:

- (a) Corresponding angles are equal in measure.

$$m\angle 1 = m\angle 5 \quad m\angle 3 = m\angle 7$$

$$m\angle 2 = m\angle 6 \quad m\angle 4 = m\angle 8$$

- (b) Alternate interior angles are equal in measure.

$$m\angle 3 = m\angle 5 \quad m\angle 4 = m\angle 6$$

- (c) Alternate exterior angles are equal in measure.

$$m\angle 1 = m\angle 7 \quad m\angle 2 = m\angle 8$$

- (d) Interior angles on the same side of the transversal are supplementary.

$$m\angle 3 + m\angle 6 = 180$$

$$m\angle 4 + m\angle 5 = 180$$

- (e) Exterior angles on the same side of the transversal are supplementary.

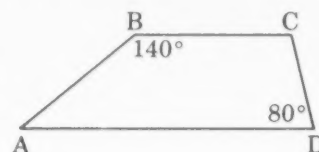
$$m\angle 1 + m\angle 8 = 180$$

$$m\angle 2 + m\angle 7 = 180$$

Given two parallel lines cut by a transversal, and the measure of any one angle, the students should be able to find the measure of the remaining seven angles based on the relations.

Applications may also include quadrilaterals that have parallel sides.

Example: In trapezoid $ABCD$, $\overline{AD} \parallel \overline{BC}$, $m\angle B = 140$, and $m\angle D = 80$. (a) Find $m\angle A$. (b) Find $m\angle C$.



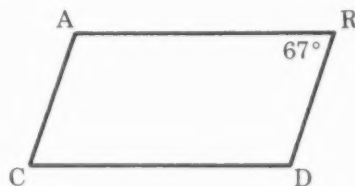
Answer: (a) 40 (b) 100

Example: In parallelogram $CARD$, $m\angle R = 67$. Find $m\angle C$.

Strategy: Draw and label parallelogram $CARD$.

Use $\overline{AR} \parallel \overline{CD}$ with transversal $\overline{RD}$ to determine $m\angle D = 113$.

Use $\overline{RD} \parallel \overline{AC}$ with transversal $\overline{CD}$ to determine $m\angle C = 67$.



Answer: 67

Have the students apply the strategy above to a general case by using algebraic terms in place of specific angle measures:



In essence, this is an algebraic proof of the statement:

"Opposite angles of a parallelogram are equal in measure."

4. Polygons

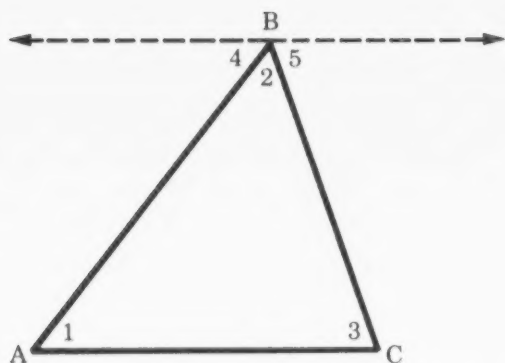
a. Angle measures of polygons

Recall that a *polygon* is a simple closed curve formed by line segments, and review the names of polygons.

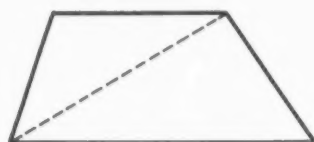
(1) Sum of interior angle measures

In grade 7, the students discovered through measurements that the *sum of the angle measures of a triangle is 180*. This concept can now be presented in a more formal manner:

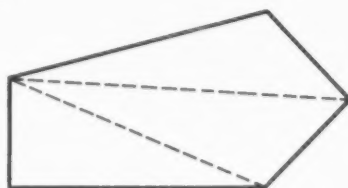
Step 1 For any triangle ABC , a line can be drawn through vertex B that is parallel to $\overline{AC}$, the base of the triangle. Label the angles in the diagram.



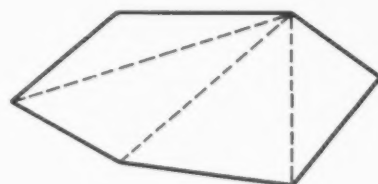
Step 2 Because a straight angle exists at B, $m\angle 4 + m\angle 2 + m\angle 5 = 180$.



Quadrilateral
 $2(180^\circ) = 360^\circ$



Pentagon
 $3(180^\circ) = 540^\circ$



Hexagon
 $4(180^\circ) = 720^\circ$

The students should recognize the pattern and discover the formula for the *sum of the angle measures of a polygon of n sides*:

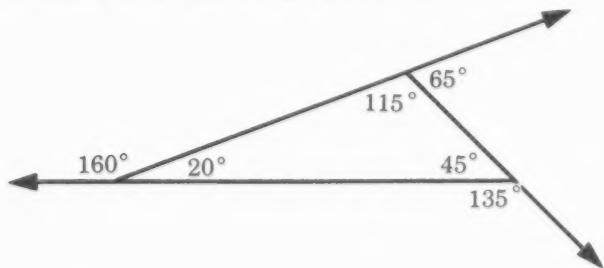
$$(n - 2) 180^\circ = S$$

Specific angle sums can be verified by the use of protractors.

(2) Regular polygons

A *regular polygon* is a polygon that is both equilateral and equiangular. That is, all sides have the same length, and all angles have the same measure. At times, a polygon can be equilateral but not equiangular, as in a rhombus. It is also possible for a polygon to be equiangular but not equilateral, as in a rectangle. The only quadrilateral that is a regular polygon is a square.

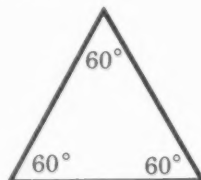
Each angle formed by adjacent sides of a polygon is called an *interior angle*. The measure of a single interior angle of a regular polygon follows directly from the formula for the sum of the angle measures.



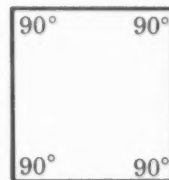
Step 3 Since parallel lines are cut by a transversal, it follows that alternate interior angles are equal in measure. Thus, $m\angle 4 = m\angle 1$ and $m\angle 5 = m\angle 3$.

Step 4 By substituting these values into the statement in step 2:
 $m\angle 1 + m\angle 2 + m\angle 3 = 180$.

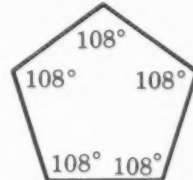
A line segment that joins two nonadjacent vertices of a polygon is called a *diagonal*. For each polygon of four or more sides, select a single vertex and draw all possible diagonals from that vertex. Relate the angle measures of the triangles formed to the angle measures of the polygons.



Sum = 180°
 $180^\circ \div 3 = 60^\circ$



Sum = 360°
 $360^\circ \div 4 = 90^\circ$

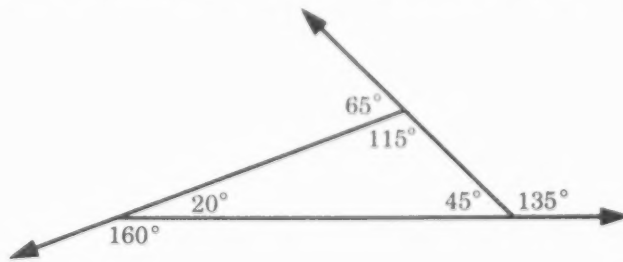


Sum = 540°
 $540^\circ \div 5 = 108^\circ$

In general terms, the measure of an interior angle of a regular polygon of n sides is $\frac{(n-2)180^\circ}{n}$

(3) Optional: Exterior angles of polygons

If a side of a polygon is extended, the angle formed by the extension and by an adjacent side of the polygon is called an *exterior angle*. Indicate alternative ways to draw an exterior angle at a vertex of a polygon, but restrict each vertex to a single exterior angle at any one time.



The study of exterior angles should then be extended to polygons of four or more sides. The students should discover that the *sum of the measures of the exterior angles of any polygon is always 360°* . This information provides another strategy for finding the measure of an interior angle of a regular polygon.

b. Triangles and quadrilaterals

The students should recall the classification of triangles according to their angles (acute triangle, right triangle, obtuse triangle, equiangular triangle) and according to their sides (scalene triangle, isosceles triangle, equilateral triangle). Definitions for specific quadrilaterals (trapezoid, parallelogram, rhombus, rectangle, square) should also be reviewed.

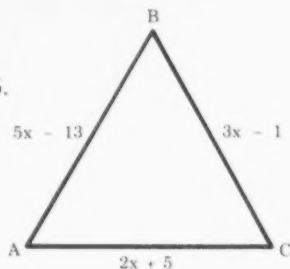
Properties related to these figures are best reviewed in problem-solving situations where many, but not all, of the solutions are based on algebraic techniques.

Example: In equilateral triangle ABC, $AB = 5x - 13$, $BC = 3x - 1$, and $AC = 2x + 5$. Find x .

Answer: 6

(Note that any of three equations could have been used: $5x - 13 = 3x - 1$, or $5x - 13 = 2x + 5$, or $3x - 1 = 2x + 5$.)

When $x = 6$, show that $AB = BC = AC = 17$.



Example: In a certain right triangle, the measures of the acute angles are in a ratio of 2:3. Find the measures of all three angles of the triangle.

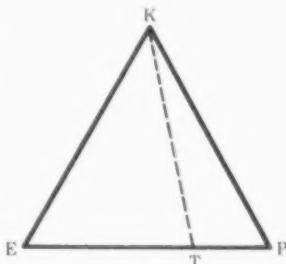
Answer: 36° , 54° , and 90°

(Use $2x + 3x + 90 = 180$, or simply $2x + 3x = 90$.)

Example: Triangle EPK is equiangular; T is any point on $\overline{EP}$; and, $\overline{KT}$ is drawn. What is the longest side of triangle EKT? (Hint: In a triangle, the longest side is opposite the greatest angle.)

Answer: $\overline{EK}$

(Reason that $m\angle E = 60^\circ$, $m\angle EKT$ is less than 60° , and $m\angle KTE$ must be greater than 60° .)



Example: In parallelogram ABCD, $m\angle A = 4y - 10$ and $m\angle C = 2y + 40$. Parallelogram ABCD must be a (a) square (b) rhombus (c) rectangle (d) trapezoid

Answer: (c) rectangle

Example: The base of a rectangular door must be sanded to fit a doorway with a warped frame. Assume this base is sanded on a straight line.

(a) If the angle at the lower left of the door is sanded to measure 92° , what is the measure of the angle at the lower right of the door?

(b) Since the door is no longer a rectangle, what type of quadrilateral is it?

Answer: (a) 88°

(b) a trapezoid

5. Geometric Solids

The concept of a geometric solid may be quite different from most students' concept of a physical solid. A *geometric solid* is any *closed surface* that completely encloses a portion of space. A geometric solid has area, but it does not have volume. Any reference to the volume of a solid actually means the volume of the interior of the solid.

Compare the above concept to a circle, a set of points in a plane at a given distance from a given point called the center. The circle is a curved line that has a length but no area. Any reference to the area of a circle actually means the area of the region bounded by the circle.

In other words, a milk container is a geometric solid, but this solid consists only of the container and not of its contents.

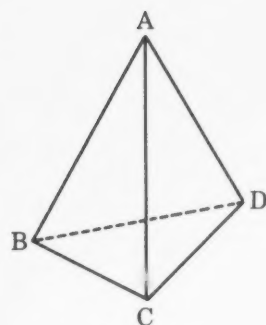
Geometric solids may have flat surfaces or curved surfaces.

a. Polyhedrons

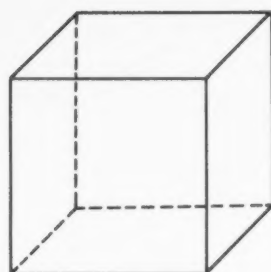
A *polyhedron* is a geometric solid that consists of flat surfaces only, where each surface is a polygon. Examples of polyhedrons (or polyhedra) include both prisms and pyramids, to be discussed later in this section.

(1) Faces, edges, and vertices

Each flat surface of a polyhedron is called a *face*. The intersection of two of these faces is called an *edge*, and the endpoints of edges are *vertices* of the solid. The tetrahedron and hexahedron (or cube) should be studied; both solids are commonly used in probability experiments.



Tetrahedron



Hexahedron (Cube)

The simplest polyhedron is the *tetrahedron*, which consists of four faces. Each face, as shown in the diagram above, is a triangle. The tetrahedron has six edges (shown above as $\overline{AB}$, $\overline{AC}$, $\overline{AD}$, $\overline{BC}$, $\overline{CD}$, $\overline{DB}$) and four vertices (points A, B, C, D).

Example: For a cube (hexahedron), determine the number of faces, the number of edges, and the number of vertices.

Answer: 6 faces, 12 edges, and 8 vertices.

(2) Optional: Convex Polyhedrons and Regular Polyhedrons

The names of polyhedrons beyond those previously mentioned, the definitions of convex polyhedrons and regular polyhedrons, and Euler's Polyhedron Theorem are topics for enrichment and investigation.

Polyhedron is derived from the Greek: poly, meaning "many" and -hedron, meaning "faces." Polyhedrons are classified according to their number of faces. In most cases, the prefixes used to name polygons are also used to name polyhedrons. Some polyhedrons include:

tetrahedron	4 faces	octahedron	8 faces
pentahedron	5 faces	decahedron	10 faces
hexahedron (cube)	6 faces	dodecahedron	12 faces
heptahedron	7 faces	icosahedron	20 faces

If a line segment joining any two points of a polyhedron contains only points inside the polyhedron, then the *polyhedron* is *convex*.

The students may investigate Euler's Theorem for convex polyhedrons as follows:

Example: For each convex polyhedron, determine the number of faces, the number of vertices, and the number of edges. If possible, find a relation or formula that is true for all polyhedrons shown.



Fig A



Fig. B

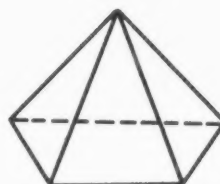


Fig. C

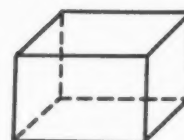


Fig. D

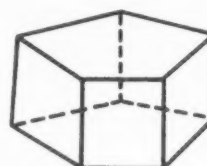


Fig. E

Answer:

Polyhedron	Faces (f)	Vertices (v)	Edges (e)
Figure A	4	4	6
Figure B	5	6	9
Figure C	5	5	8
Figure D	6	8	12
Figure E	7	10	15

Euler's Theorem: $f + v - e = 2$, or $f + v = e + 2$, or equivalent

A *regular polyhedron* is a convex polyhedron that meets two conditions: its faces are congruent regular polygons, and the same number of edges meet at each vertex. There are only five regular polyhedrons, sometimes called the Platonic solids:

regular tetrahedron - 4 faces, equilateral triangles

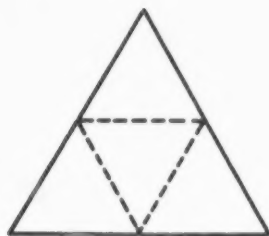
regular hexahedron (cube) - 6 faces, square

regular octahedron - 8 faces, equilateral triangles

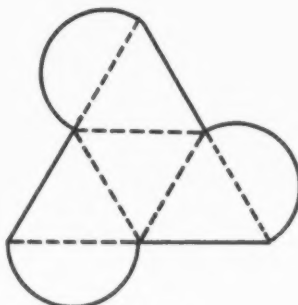
regular dodecahedron - 12 faces, regular pentagons

regular icosahedron - 20 faces, equilateral triangles

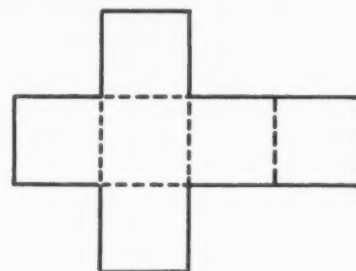
Models of the regular polyhedrons may be built by the students, using straws or nets (plane diagrams). If a net is used, have the students place tabs on every other edge so that, when the net is folded, the tabs can be glued to hold the polyhedron together.



Net for Tetrahedron



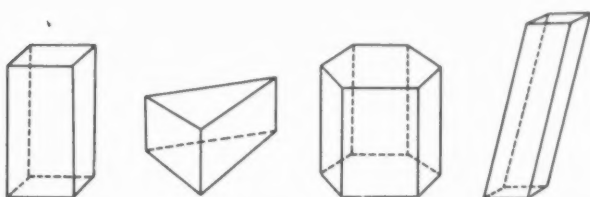
Net with Tabs



Net for Hexahedron (Cube)

b. Prisms

A *prism* is a polyhedron with two congruent and parallel faces (called bases) which are polygons. All other faces of a prism (called lateral faces) are parallelograms formed by joining corresponding vertices of the bases.



Rectangular Prism Triangular Prism Hexagonal Prism Rectangular Prism

A prism is rectangular, triangular, hexagonal, and so on according to the shape of its bases. The students should be able to identify many prisms in the world, not just the standard triangular prisms used in science classrooms.

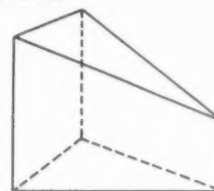
If the lateral faces are perpendicular to the bases, the prism is a *right prism*. If the lateral faces are not perpendicular to the bases, the prism is an *oblique prism*. For example, in the diagrams above, a right rectangular prism is shown at the far left; an oblique rectangular prism is displayed at the far right.

Example: How many faces of an octagonal prism are parallelograms?

Answer: 8 (the 8 lateral faces)

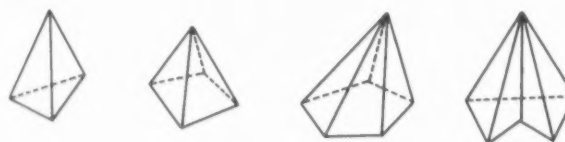
Example: Is the polyhedron shown here a prism?
Explain why or why not.

Answer: It is not a prism because it does not have two parallel bases.



c. Pyramids

A *pyramid* is a polyhedron in which one face (called the base) may be any polygon, and all other faces are triangles that meet at a common vertex.



Examples of Pyramids

The common vertex of the triangular (lateral) faces is called the *vertex* or *apex* of the pyramid. The *altitude* of a pyramid is the segment from the vertex that is perpendicular to the polygon which is the base. Emphasize that a pyramid can never be a prism, and a prism can never be a pyramid, although both are examples of polyhedrons.

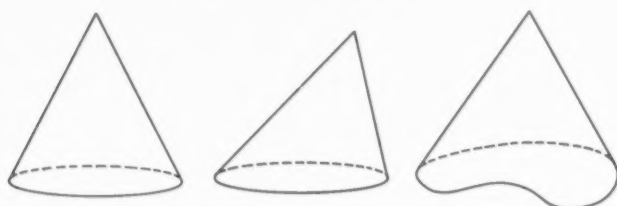
Optional: In a *regular pyramid*, the altitude connects the vertex to the center of the base, and the base is a regular polygon. If either or both of these conditions are missing, the pyramid is

called *irregular*. An enrichment project on the great pyramids of Egypt can yield a wealth of mathematics, especially after studying measurements and the Pythagorean Theorem in this strand.

d. Cones

A *cone* is a geometric solid that consists of one flat surface and one curved surface:

- (1) The flat surface, called the *base*, consists of a region bounded by a closed curve on a plane.
- (2) The curved surface is composed of a point off the plane and all possible line segments from that point to every point on the closed curve that determines the base. The point selected off the plane is called the *vertex* or *apex* of the cone.



Examples of Cones

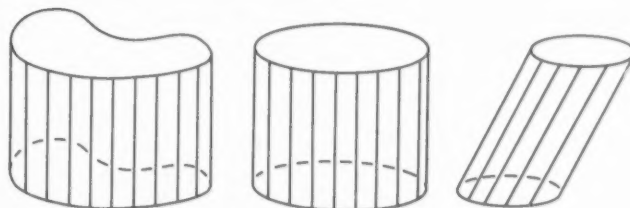
Review the definition of a circle and its parts (radius, diameter, center, chord, and arc). If a cone has a circular base, it is called a *circular cone*. However, the base of a cone can be determined by any closed curve.

The *altitude* of a cone is the segment (or its length) from the vertex that is perpendicular to the base. A cone is *not* a polyhedron.

e. Cylinders

A *cylinder* is a geometric solid that consists of two congruent flat surfaces and one curved surface:

- (1) The two flat surfaces are congruent regions that lie in parallel planes; these regions are bounded by congruent closed curves.
- (2) The curved surface is composed of all the line segments that join corresponding points of the two congruent closed curves.



Cylinder

Right
Circular
Cylinder

Oblique
Circular
Cylinder

The flat surfaces are called the *bases* of the cylinder. These bases must be determined by congruent closed curves, but they need not be circles. If the bases are circles, a *circular cylinder* exists.

A cylinder whose curved (lateral) surface is perpendicular to the bases is called a *right cylinder*. A cylinder whose curved (lateral) surface is not perpendicular to the bases is called an *oblique cylinder*.

The students will apply concepts of measurement primarily to the *right circular cylinder*; in this case, the *altitude* is a segment (or its length) perpendicular to both bases. The altitude of an oblique cylinder need not be mentioned but can be described as a segment perpendicular to the two planes that contain the bases. A cylinder is *not* a polyhedron.

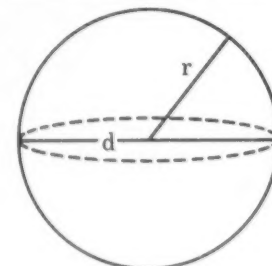
f. Spheres

A *sphere* is a geometric solid that is the set of all points at a given distance from a given point called the center of the sphere.

How does a sphere differ from a circle? What differences exist between the definition of a sphere and the definition of a circle? A sphere is a curved surface that completely encloses a portion of space. A circle is a line drawn on a surface or a plane.

The *radius* of a sphere is any line segment (or a distance) from the center to any point on the sphere. The *interior* of a sphere is the set of points whose distance from the center is less than the radius. The *exterior* of a sphere is the set of points whose distance from the center is greater than the radius.

The *diameter* of a sphere is any line segment (or a distance) determined by two endpoints on the sphere and containing the center of the sphere.



Sphere

Optional: A discussion of great circles, small circles, and hemispheres can provide the students with enrichment and areas of application.

If a plane intersects a sphere, the intersection will either be a single point or a circle. A *great circle* is the set of points in the intersection of a sphere and a plane that contains the center of the sphere. If the plane does not contain the center of the sphere but the intersection is a circle, then the intersection is called a *small circle*.

A *hemisphere* is half a sphere bounded by a great circle. These terms can be related to our planet, a globe, navigation, and aviation.

B. SYSTEMS OF MEASUREMENT

1. Metric System

a. Review equivalent measures in metric

Metric measures should appear in applications throughout the entire curriculum. Some review, however, of equivalent measures within the metric system may be necessary. Include square and cubic units of measure to prepare students for area and volume. (See *Mathematics 7 Syllabus*.)

Example: Find the numeral that makes each sentence true.

- (a) 2350g = _____ kg (b) 400 mL = _____ L
(c) 1m² = _____ cm² (d) 1 cm³ = _____ mm³

Answer: (a) 2.35 (b) 0.4 (c) 10,000 (d) 1,000

Example: Arrange the following lengths in order, from the largest to the smallest:

5.5 m, 500 cm, 550 mm, 0.05 km

Strategy: Convert the lengths to a common unit of measure; then order.

5.5 m = 5.5 m
500 cm = 5 m
550 mm = 0.55 m
0.05 km = 50 m

Answer: 0.05 km, 5.5 m, 500 cm, 550 mm

b. Optional: Measures with scientific applications

(1) Force: newton (N)

The *newton* (N) is the unit of force (push or pull) which gives a mass of 1 kilogram an acceleration of 1 meter per second, per second.

Discuss the formula $F = ma$ where:

F = force in newtons (N)

m = mass in kilograms (kg)

a = acceleration in meters/second/second (m/s/s) or (m/s²)

Applications should include the force of gravity expressed in newtons. On earth, the force of gravity on a mass of 1 kg is about 9.8N.

Example: How many newtons of force are needed to accelerate a 5 kg mass at a rate of 2m/s/s?

Answer: 10N (Use $F = ma$)

Example: Acceleration due to earth gravity is 9.8 m/s/s at the surface. How much force does a 5 kg rock exert on the ground?

Answer: 49N (Use $F = ma$)

Example: How do you determine the force of gravity between you and the earth in newtons?

Answer: Multiply the mass of the person in kg by 9.8 m/s/s. (For a student whose mass is 55 kg, the force is $55 \text{ kg} \times 9.8 \text{ m/s/s} = 539.0\text{N}$, or 539 newtons.)

Example: An acceleration of 9.8m/s/s on earth means that an object falls in earth's gravity so that its speed increases by 9.8 meters per second for each second that it is falling. The distance of this fall is the product $(\frac{1}{2})(9.8\text{m})(t^2)$, where t = time in seconds.

Time in Seconds	Speed in Meters/Second	Distance in Meters
0	0	0
1	9.8	4.9
2	19.6	19.6
3	29.4	44.1
4	39.2	78.4

Find the speed and distance fallen: (a) after 5 seconds (b) after 6 seconds

Answer: (a) 5 seconds: 49 m/s, 122.5 m
(b) 6 seconds: 58.8 m/s, 176.4 m

(2) Others

A container whose volume is one cubic centimeter holds one milliliter of water, leading to a commonly accepted interchange of measures that students may experience in a science experiment. That is, $1 \text{ cm}^3 = 1\text{mL}$. Other examples can be identified by science teachers at your school.

2. Customary System of Measurement

Review, and occasionally use in problems, those customary units of measure discussed in grade 7. Mention equivalent square and cubic units within the system, such as $1 \text{ sq.ft.} = 144 \text{ sq.in.}$, and $1 \text{ cu.yd.} = 27 \text{ cu.ft.}$

Conversions from one unit of measure to another should essentially be kept *within* a single system of measurement. If teachers of science, home economics, industrial arts, or other subjects require that the students make con-

versions between metric and customary measures, then the students should be given conversion charts and/or calculators to assist them in this task.

C. MEASUREMENT OF GEOMETRIC FIGURES

1. Perimeter

a. Polygons

The *perimeter of a polygon* is the sum of the lengths of its sides. Standard applications of perimeter should be kept to a minimum, but the topic remains a good vehicle for problem solving. It is suggested that algebraic solutions be required in a number of exercises.

Optional: Perimeters of regular polygons

Example: The length of a rectangle is 4 cm more than its width. If the perimeter of the rectangle is 40 cm, find its dimensions.

Let w = width; then $w + 4$ = length
Solve $w + w + w + 4 + w + 4 = 40$.
or $2(w + w + 4) = 40$

Answer: 8 cm by 12 cm

Example: Each side of a regular octagon measures $\frac{5}{8}$ yard, and each side of a regular hexagon measures 3 feet 4 inches. Which polygon has a larger perimeter?

Answer: Neither; each polygon has a perimeter of 20 feet.

b. Circles

Circumference may be treated briefly here since this topic will reappear throughout this strand. In grade 7, the students explored the ratio $C/d = \pi$ and used rational number approximations of π to perform computations:

$$\pi \approx 3.14 \quad \pi \approx \frac{22}{7} \quad \pi \approx 3.1416$$

Solutions to some problems may now be left in terms of π . If k is a rational number, then $k\pi$ is irrational and hence a real number.

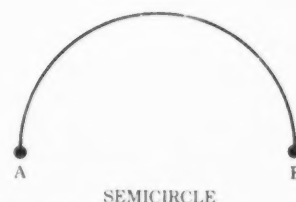
Find the lengths of arcs, specifically semicircles and quarter-circles, which will be used to form composite shapes in the next section.

Example: Express, in terms of π , the circumference of a circle whose radius is 7.

Answer: 14π

Example: A *semicircle* is one-half of a circle. If the

straight line distance from A to B is 14 mm, find the length of the semicircle AB to the nearest millimeter.



Using $\pi = \frac{22}{7}$

$$C = \pi d = \frac{22}{7} \times 14 = 44$$

$$\text{length of } \widehat{AB} = \frac{1}{2} \times 44 = 22$$

Answer: 22 mm

Using $\pi = 3.14$

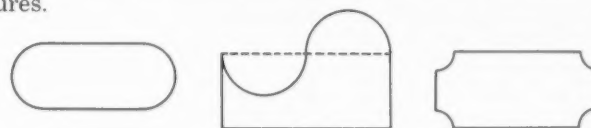
$$C = \pi d = 3.14 \times 14 = 43.96$$

$$\text{length of } \widehat{AB} = \frac{1}{2} \times 43.96 = 21.98$$

21.98 is rounded to 22

c. Composite Shapes

Many plane figures in our world are *composite shapes* consisting of straight line segments and curves. The students should find the perimeters of a variety of such figures.



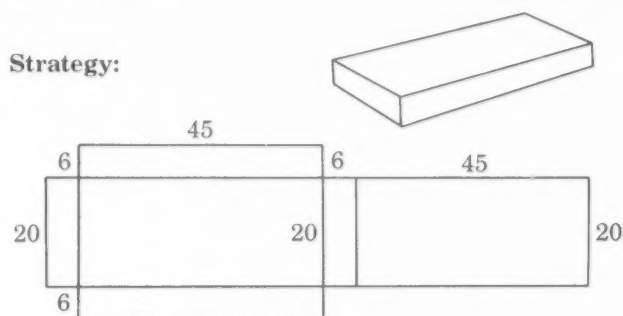
2. Surface Area

Given the necessary measurements of a geometric solid, the students should be able to find the *surface area* of the following: a *rectangular prism*, a *cube*, and a *right circular cylinder*. In the first two cases, the surface area can be displayed by a net which shows the solid in a flattened form. A physical model, such as a tin can, can be taken apart to demonstrate how the surface area of a right circular cylinder is obtained.

a. Rectangular Prisms and Cubes

Example: Find the surface area of a gift box, 45 cm by 20 cm by 6 cm.

Strategy:



$$2 \text{ faces, each } 6 \text{ by } 20: 2 \times 6 \times 20 = 240$$

$$2 \text{ faces, each } 6 \text{ by } 45: 2 \times 6 \times 45 = 540$$

$$2 \text{ faces, each } 20 \text{ by } 45: 2 \times 20 \times 45 = +1800$$

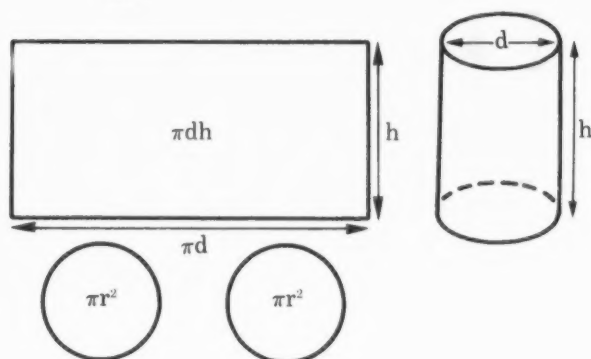
$$\hline 2580$$

Answer: 2580 cm²

b. Right Circular Cylinders

Example: A brand of oatmeal is packaged in a right circular cylinder whose height is 7 inches and whose diameter is 4 inches. Find the surface area of the container to the nearest square inch. (Use $\pi = 3.14$)

Strategy: The two bases are circles with radius = 2 inches. The lateral surface is a rectangle whose length is the circumference of the circle.



$$\text{Top base: } \pi r^2 = 3.14 \times 2 \times 2 = 12.56$$

$$\text{Bottom base: } \pi r^2 = 3.14 \times 2 \times 2 = 12.56$$

$$\text{Lateral Surface: } \pi dh = 3.14 \times 4 \times 7 = + 87.92$$

$$\hline 113.04$$

Answer: 113 sq.in.; to the nearest square inch, 113

An alternative strategy for the problem just solved is one in which the surface area is expressed in terms of π . After the exact measurement is found, a rational approximation can be determined from the solution.

$$\begin{aligned} \text{Surface Area} &= \pi r^2 + \pi r^2 + \pi dh \\ &= \pi \times 2 \times 2 + \pi \times 2 \times 2 + \pi \times 4 \times 7 \\ &= 4\pi + 4\pi + 28\pi \\ &= 36\pi \end{aligned}$$

Area of Rectangle



$$\begin{aligned} B &= lw \\ &= 2 \times 4 \\ &= 8 \text{ sq. units} \end{aligned}$$

Then, surface area = $36\pi = 36 \times 3.14 = 113.04$, or 113 to the nearest square inch.

Optional: If the surface area of any other geometric solid (such as a sphere) is to be found, the students should be provided with the appropriate formula.

3. Volume

Given the necessary measurements of a geometric solid, the students should be able to find the *volume* of a *rectangular prism*, a *cube*, a *triangular prism*, and a *right circular cylinder*.

The procedures for finding the volume of a pyramid, a cone, and a sphere may be demonstrated in class, but the students should be given appropriate formulas when asked to find the volumes of these solids.

Since a geometric solid is actually a surface, any reference to the volume of a solid indicates the volume of the interior or the space bounded by that solid. Volume is a measure in cubic units.

a. Prisms

In grade 7, the volume of a *rectangular prism* was studied using length l , width w , height h , and the formula $V = lwh$. As a special case, the volume of a *cube* with edge e was also studied, using the derived formula $V = e^3$.

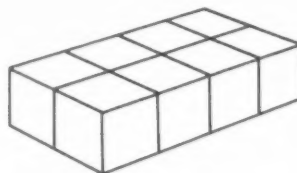
Physical models, including boxes filled with cubes, are helpful in developing the concept of volume. The volume of right prisms must be studied, although bases may be limited to triangles and rectangles. The volume of oblique prisms is **optional**.

Discuss the difference between area and volume, and between square units and cubic units of measure. Develop the general equation for the volume of a prism:

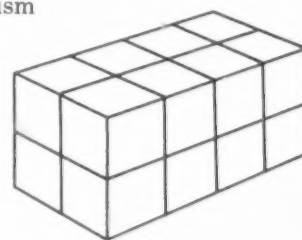
$$V = Bh$$

where B represents the area of the base, and h represents the altitude of the prism. This formula is valid for all prisms, both right and oblique.

Volume of Rectangular Prism

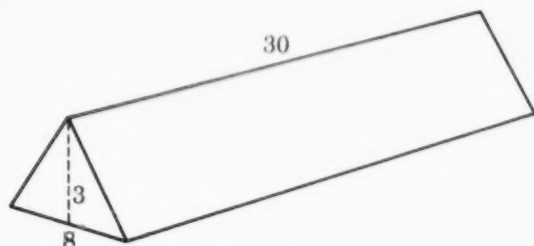


$$\begin{aligned} V &= Bh \\ &= 8 \times 1 \\ &= 8 \text{ cu. units} \end{aligned}$$



$$\begin{aligned} V &= Bh \\ &= 8 \times 2 \\ &= 16 \text{ cu. units} \end{aligned}$$

Example: The lengths in the accompanying diagram are given in centimeters. Find the volume of this triangular prism.



Area of base : $B = \frac{1}{2} \times 3 \times 8 = 12 \text{ cm}^2$

Volume of prism: $V = Bh = 12 \times 30 = 360 \text{ cm}^3$

Answer: 360 cm^3

Example: A package of granola bars is 7" by 6" by $1\frac{1}{2}$."

- Find the volume in cubic inches for this package.
- Find the volume of a cube whose edge measures 4".
- Find the surface area of the granola package.
- Find the surface area of the cube.
- What are some advantages and disadvantages of the present granola package? Should the granola bars be put into a cube-like package? Why?

Answer: (a) 63 cu.in. (b) 64 cu.in.

(c) 123 sq.in. (d) 96 sq.in.

(e) Answers vary: The present granola package is more costly because it requires more paper (greater surface area); however, this allows the company more room to advertise and show its product, etc.

b. Right circular cylinders

The formula for the volume of a right circular cylinder is developed from the general formula, $V = Bh$. Here, the area of the base is the area of a circular region, or $B = \pi r^2$. This leads to the volume formula:

$$V = \pi r^2 h$$

where r is the radius of the circular base and h is the altitude of the cylinder.

Consider the oatmeal package discussed earlier. Its radius is 2 inches, and its height is 7 inches. Its volume is $\pi r^2 h = \pi \times 2 \times 2 \times 7 = 28\pi$ square inches, or its appropriate rational approximation.

c. Pyramids

The volume of a pyramid is best discovered through experimentation. Select an open prism and an open pyramid whose bases are congruent and whose altitudes are equal in measure. Fill the pyramid with water or sand,

and then pour these contents into the prism. Since the contents of exactly 3 pyramids are needed to fill the prism, the volume of the pyramid is one-third the volume of the prism. Therefore, the formula for the volume of a pyramid is:

$$V = \frac{1}{3} Bh$$

where B represents the area of the base and h represents the height of the pyramid.

d. Cones

Experimentation procedures similar to those used with pyramids and prisms can be applied to cones and cylinders. Select an open cone and an open cylinder whose bases are congruent and whose altitudes are equal in measure. Since the contents of exactly 3 cones will fill the cylinder, the volume of the cone is found by the formula:

$$V = \frac{1}{3} Bh \quad \text{or} \quad V = \frac{1}{3} \pi r^2 h$$

where r represents the radius of the circular base, and h represents the height of the cone.

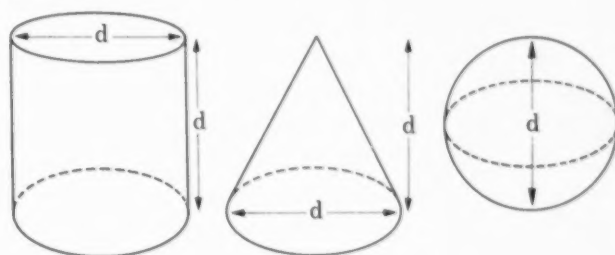
Example: A paper cup shaped like a cone has a circular base with a diameter of 6 cm and a height of 8 cm. Find its volume in cubic centimeters.

$$V = \frac{1}{3} \pi r^2 h = \frac{1}{3} \times \pi \times 3 \times 3 \times 8 = 24\pi$$

Answer: 24π , or a rational approximation

e. Spheres

Consider the experiment described above for the cone and the cylinder. Models are available where the diameter of each circular base is equal in length to the altitude of the cone and the cylinder. In the set below, a sphere has a diameter equal to the diameters of the cone and of the cylinder.



If the cylinder is filled with water and the sphere is pushed into the cylinder, the sphere will displace an amount of water. Remove the sphere from the cylinder and pour the remaining water from the cylinder into the cone; the water should fill the cone exactly. Since the volume of the cone is $\frac{1}{3} Bh$, the volume of the sphere must be $\frac{2}{3} Bh$. But here, $B = \pi r^2$ and $h = d = 2r$.

$$\text{Volume of sphere} = \frac{2}{3} Bh = \frac{2}{3} \pi r^2 \times 2r = \frac{4}{3} \pi r^3$$

If the students are asked to find the volume of a sphere, they should be given the formula as an aid:

$$V = \frac{4}{3} \pi r^3$$

Example: The diameter of a ball bearing is 6 millimeters.

(a) Find in terms of π the volume of this ball bearing.

(b) Find a rational approximation of the volume, to the nearest cubic millimeter.

$$V = \frac{4}{3} \pi r^3 = \frac{4}{3} \times \pi \times 3 \times 3 \times 3 = 36\pi$$

$$36\pi = 36 \times 3.14 = 113.04, \text{ or } 113 \text{ to the nearest cubic millimeter}$$

Answer: (a) $36\pi \text{ mm}^3$ (b) 113 mm^3

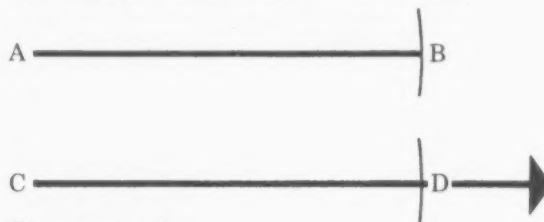
D. CONSTRUCTIONS AND ACTIVITIES

1. Constructions

In an experiment, tools of measure such as a ruler and a protractor are used. Constructions, however, are performed with straight edge and compasses only.

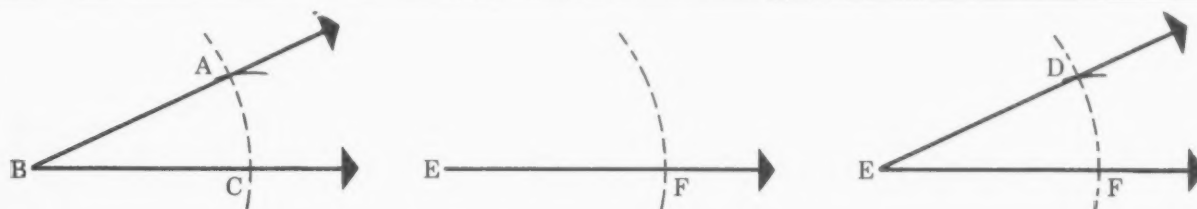
a. Copy a line segment.

Given $\overline{AB}$, construct $\overline{CD}$ where $CD = AB$.



b. Copy an angle.

Given $\angle ABC$, construct $\angle DEF$ so that the two angles are equal in measure.



From B, swing an arc and treat A and C as fixed points on the rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$. Mark $\overline{EF}$ on the ray $\overrightarrow{EF}$ so that $EF = BC$. Swing an arc from C to establish radius AC. Then swing an arc from F to determine radius FD where $FD = CA$. The point of intersection of the arcs is D. Draw $\overline{ED}$.

2. Activities Involving Constructions

Various activities, based on the given constructions, should follow. After the constructions are performed in an

activity, the students may use tools of measure in some cases to verify their conjectures.

Activity: Draw an acute angle. Using compasses and straight edge only, construct an angle whose measure is twice the measure of the original angle.

Activity: Construct a triangle in which two angles have the same measure.

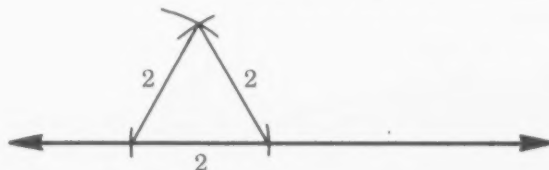
Activity: Three segments are given whose lengths are 2, 3, and 5 centimeters.



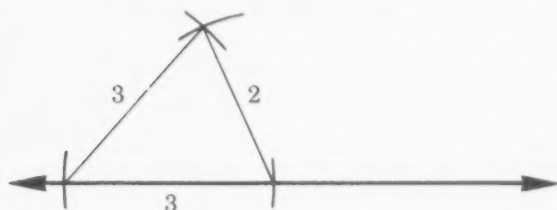
Using any of the given lengths as many times as you wish:

- How many equilateral triangles can be constructed? Construct one.
- How many isosceles triangles can be constructed? Construct one.
- Is it possible to construct a triangle whose sides measure 2 cm, 3 cm, and 5 cm? Or a triangle of 2 cm, 2 cm, and 5 cm? Why? Why not?

Answer: (a) Three equilateral triangles can be constructed: 2-2-2, 3-3-3, 5-5-5.



- (b) Five isosceles triangles can be constructed: 5-5-3, 5-5-2, 3-3-5, 3-3-2, 2-2-3.



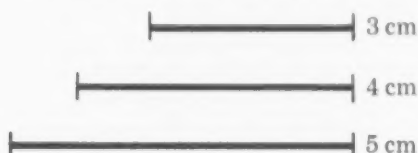
- (c) It is not possible to construct such triangles. In every triangle, the sum of the lengths of the two smaller sides must be greater than the length of the largest side.

E. THE PYTHAGOREAN THEOREM

1. Experiments with Triangles

Allow time for the students to experiment with triangles, using both tools of construction and tools of measurement, before stating the Pythagorean Theorem.

Activity: Three segments are given whose lengths are 3, 4, and 5 centimeters.



- Using compasses only, construct a triangle whose sides measure 3 cm, 4 cm, and 5 cm.
- Use a protractor to find the measure of the largest angle of the triangle.
- What type of triangle has been constructed?
- How is the sum of 3^2 and 4^2 related to 5^2 ?

Answer: (a) construction (b) 90° (c) A right triangle, or a right scalene triangle
(d) $3^2 + 4^2 = 5^2$

Activity: Repeat the experiment with segments of 5, 12, and 13 centimeters.

Answer: A right triangle is constructed, and $5^2 + 12^2 = 13^2$

Activity: Repeat the experiment with segments of 4, 5, 7 centimeters.

Answer: A right triangle is *not* formed (the triangle is obtuse), and $4^2 + 5^2 \neq 7^2$. Here, $4^2 + 5^2 < 7^2$.

2. Stating the Pythagorean Theorem

In a right triangle, the largest side (which is opposite the 90° angle) is called the *hypotenuse*. The two remaining sides are called the *legs* of the triangle. By convention, a and b represent the lengths of the legs of the triangle, and c represents the length of the hypotenuse.

The *Pythagorean Theorem*, which is valid for right triangles only, states that the sum of the squares of the lengths of the two legs is equal to the square of the length of the hypotenuse. In symbols:

$$a^2 + b^2 = c^2$$

There are many proofs of the Pythagorean Theorem, but the presentation of any such proof is not required at this grade level.

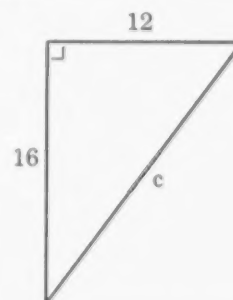
3. Applications

Present problems in which the lengths of two sides of a right triangle are given, and the Pythagorean Theorem is used to find the length of the third side. Indicate that these problems are illustrative of indirect measure. In initial exercises, restrict lengths to whole number measures only. Eventually include lengths that are not whole numbers, both rational and irrational. In some exercises, express rational approximations for irrational values; the procedures to find these approximations are discussed in the Number Concepts strand.

Example: Mr. Modell drove his car 16 km north and then 12 km east. How far is he from his starting point?

Strategy: Draw a diagram, and label all parts. Apply the Pythagorean Theorem.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 16^2 + 12^2 &= c^2 \\ 256 + 144 &= c^2 \\ 400 &= c^2 \\ \pm \sqrt{400} &= c \\ \pm 20 &= c \end{aligned}$$



Since a length is positive, disregard the negative value.

Answer: 20 km

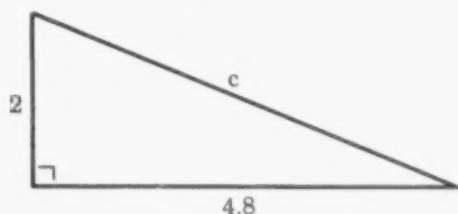
Example: A gutter on Carole Henney's house was filled with leaves and needed cleaning. To reach the gutter, Carole used a 25-ft. ladder. She placed the ladder so

that the bottom was 7 feet from the base of the house and its top just touched the gutter. How high off the ground is the gutter?

$$\begin{aligned}a^2 + b^2 &= c^2 \\7^2 + b^2 &= 25^2 \\49 + b^2 &= 625 \\b^2 &= 576 \\b &= \pm 24 \text{ (Disregard negative)}\end{aligned}$$

Answer: 24 feet

Example: The legs of a right triangle measure 4.8 and 2. Find the length of the hypotenuse.

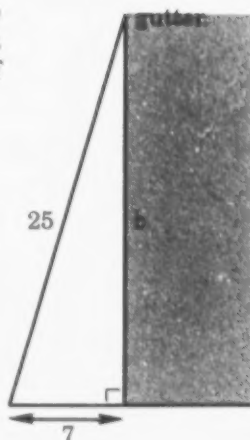
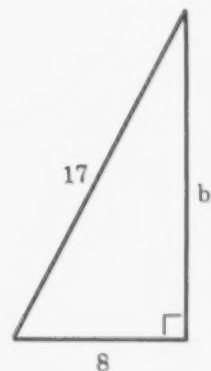


Answer: 5.2

Example: In a certain right triangle, the length of one leg is 8 cm, and the length of the hypotenuse is 17 cm.

- Find the length of the remaining leg.
- Find the perimeter of the triangle.
- Find the area of the triangle.

Answer: (a) 15 cm (b) 40 cm (c) 60 cm²



Example: If the lengths of two sides of a right triangle are 1 and 2, what is the length of the third side?

Strategy: The length of 2 could be the length of a leg or the hypotenuse.

From $1^2 + 2^2 = c^2$, it follows that $c = \sqrt{5}$.

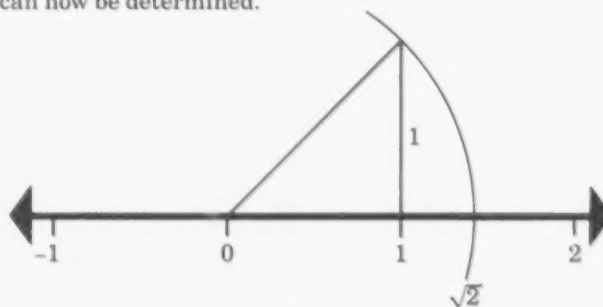
From $1^2 + b^2 = 2^2$, it follows that $b = \sqrt{3}$.

Answer: $\sqrt{3}$ or $\sqrt{5}$

A rational approximation could be given for each irrational number, namely, $\sqrt{3} \approx 1.7$ and $\sqrt{5} \approx 2.2$ (to the nearest tenth.)

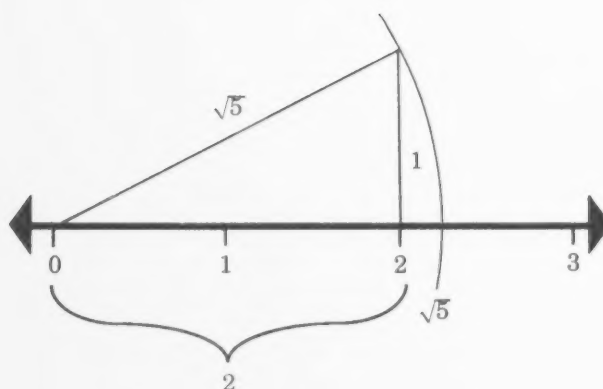
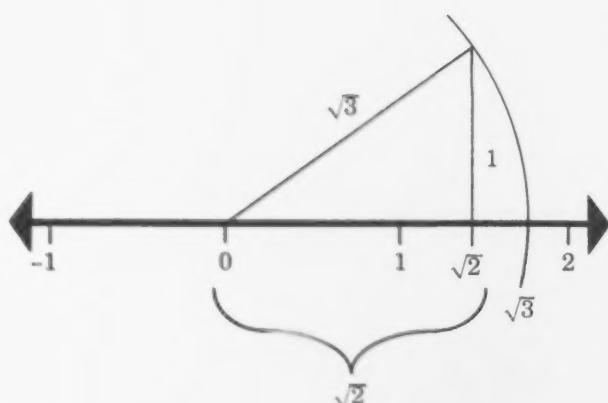
4. Optional: Constructing Irrational Lengths

In the Number Concepts strand, the students approximated the placement of points on the real number line that corresponded to irrational numbers. Demonstrate how the exact location of such points on the number line can now be determined.



In the diagram above, draw a perpendicular segment of length 1 at the point corresponding to 1 on the number line. Complete the triangle as shown. By the Pythagorean Theorem, the hypotenuse has a measure of $\sqrt{2}$. Place the point of the compasses on 0 and swing an arc of radius $\sqrt{2}$ to locate the exact point that corresponds to $\sqrt{2}$ on the real number line.

This process can be continued to locate the points that correspond to other irrational numbers. In the diagrams of right triangles that follow, when the legs measure $\sqrt{2}$ and $\sqrt{1}$, the measure of the hypotenuse is $\sqrt{3}$. Also, when the legs measure 2 and 1, the measure of the hypotenuse is $\sqrt{5}$.



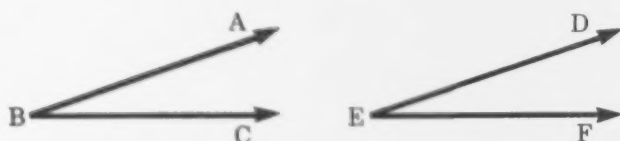
F. CONGRUENCE

1. Line Segments and Angles

When two line segments have the same measure, or when two angles have the same measure, they are said to be *congruent*. Throughout this curriculum, references to equal measures could have been replaced with the word "congruent" (in symbols: $\cong$).

A $\bullet$ — $\bullet$ B AB is equal in measure (or length) to CD: $AB = CD$

C $\bullet$ — $\bullet$ D Segment AB is congruent to segment CD: $\overline{AB} \cong \overline{CD}$



Angle ABC is equal in measure to angle DEF:
 $m\angle ABC = m\angle DEF$

Angle ABC is congruent to angle DEF:
 $\angle ABC \cong \angle DEF$

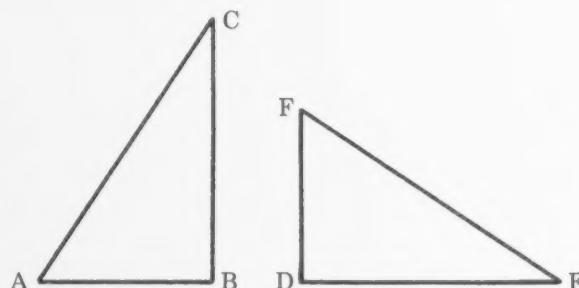
2. Congruent Figures

Plane figures are *congruent* if they have the same shape and the same size. In other words, the figures can be made to coincide.

Polygons are *congruent* if and only if their corresponding sides are equal in measure and their corresponding angles are equal in measure. In other words, the corresponding parts of congruent polygons must be congruent.

The students should be able to identify *corresponding* parts of congruent polygons and, where necessary, use tools of measure to verify that these parts are truly congruent.

Example: The triangles in the accompanying diagram are congruent. Identify all corresponding parts, using statements of congruence.



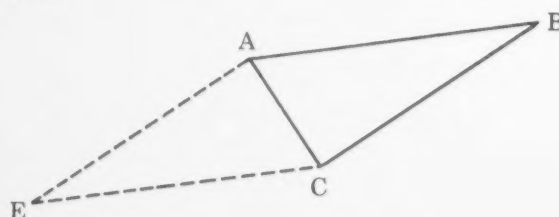
Answer: $\angle A \cong \angle F$ $\overline{AB} \cong \overline{FD}$
 $\angle B \cong \angle D$ $\overline{BC} \cong \overline{DE}$
 $\angle C \cong \angle E$ $\overline{AC} \cong \overline{FE}$

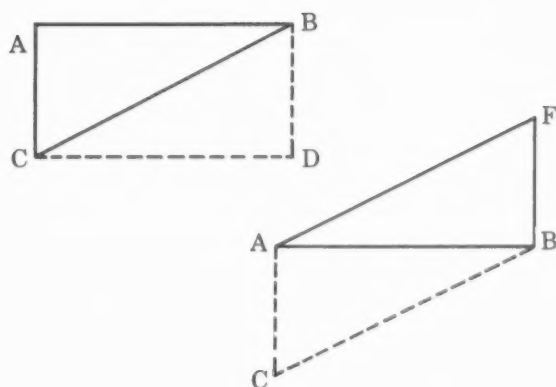
Example: If two congruent triangles are connected so that one pair of their corresponding sides coincides, a parallelogram is formed.

(a) Draw any triangle ABC. Then, using compasses and a straight edge, copy the triangle to form a parallelogram.



(b) How many different parallelograms can be constructed?



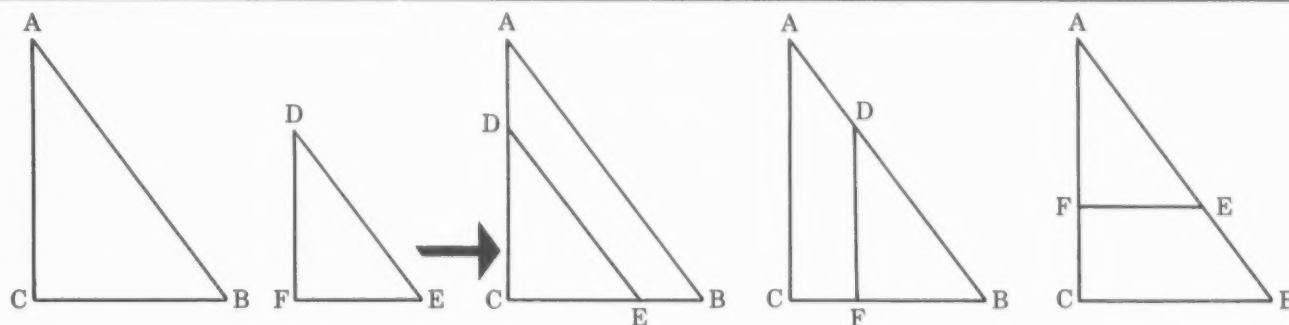


Answer: (a) construction (b) 3

Additional practice can be provided in determining whether or not two polygons (usually triangles) are congruent, and in identifying corresponding parts of congruent polygons. Suggested activities include drawing diagonals in given quadrilaterals, drawing altitudes in given triangles, and examining nets of solid figures.

G. SIMILARITY

In the strand on Ratio, Proportion and Percent, the students were asked to solve simple geometric applications using proportions. Similar polygons were treated like scale drawings, and overlapping figures were avoided.



Select any model and practice writing proportions using the given lengths. In the figure above $AC/CB = DC/CE$ and $DE/EC = AB/BC$ etc.

Example: Tell whether each statement is true or false.

- Any two right triangles are similar.
- Any two squares are similar.
- Any two rectangles are similar.
- All similar figures are congruent.
- All congruent figures are similar.

The exercises that follow here may be integrated in the earlier unit.

1. Similar Figures

If two plane figures have the same shape but not necessarily the same size, they are said to be *similar figures*. Photographic enlargements, scale drawings, and maps provide applications of similarity in our world.

Polygons are similar if and only if:

- All pairs of corresponding angles are congruent, and
- All pairs of corresponding sides have lengths that are in proportion. (That is, the ratios of the lengths of corresponding sides are equal.)

The symbol $\sim$ means "is similar to."

To experience similar polygons in a concrete way, perform one or both of the following activities.

Activity: Ask each student in the class to draw a triangle, using protractor and straight edge, so that two of its angles measure 90° and 30° . Compare the various triangles drawn, and use the definition to show why the triangles are similar.

Activity: Cut two similar triangles out of cardboard. Use these models to show the figures formed by allowing the three pairs of corresponding angles to coincide.

Answer: (a) False (b) True (c) False
(d) False (e) True. The ratio of the lengths of corresponding sides is 1:1.

2. Applications Using Proportions

Many problems involving similar polygons are solved by handling proportions in a numerical or algebraic mode. Similar figures need not be separated nor displayed in the same orientation.

Example: To determine AB, the distance across a lake, a scout paced out distances: BC = 50 meters and CD = 15 meters. Both $\angle ABC$ and $\angle BCD$ are right angles. With the aid of a transit, the scout sighted point A from point D, found point E, and measured the distances: BE = 40 meters and EC = 10 meters. What is AB, the distance across the lake?

Strategy: Establish that the triangles are similar by showing that two pairs of corresponding angles are congruent. (It follows that the remaining or third angles must also be congruent.)

$\angle ABE \cong \angle DCE$, since both are given as right angles.

$\angle BEA \cong \angle CED$, since vertical angles are congruent.

Since the triangles are similar, the lengths of their corresponding sides are in proportion. Write and solve an equation involving AB.

$$AB:BE = DC:CE$$

$$x:40 = 15:10$$

$$10x = 600$$

$$x = 60$$

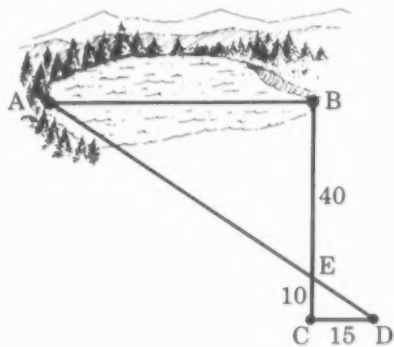
or

$$\frac{AB}{BE} = \frac{DC}{CE}$$

$$\frac{x}{40} = \frac{15}{10}$$

$$x = 60$$

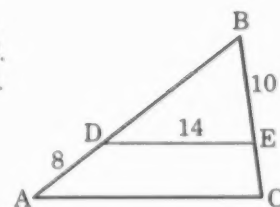
Answer: 60 meters



(a) If AD = 8, find DB.

(b) If BE = 10, find EC.

(c) If DE = 14, find AC.



Answer: (a) 16 (b) 5 (c) 21

(Note that $DE:AC = 2:3$)

3. Optional: Ratio of Perimeters and Ratio of Areas

For similar polygons, investigate the ratio of perimeters and the ratio of areas. If the polygons are limited to triangles and standard quadrilaterals, the students may use rectangular coordinate paper as an aid. Here, sides may be counted in finding perimeter, and squares may be counted as units of area. Help the students to discover the following pattern for similar polygons:

Ratio of lengths of
corresponding sides

a:b

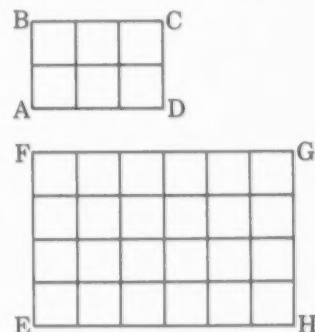
Ratio of
perimeters

a:b

Ratio
of areas

$a^2:b^2$

Example: Rectangle ABCD ~ rectangle EFGH, as given by the diagrams. Find the ratios in simplest form: (a) AB:EF (b) BC:FG (c) perimeter of ABCD: perimeter of EFGH (d) area of ABCD: area of EFGH.



Answer: (a) $2:4 = 1:2$ (b) $3:6 = 1:2$
(c) $10:20 = 1:2$ (d) $6:24 = 1:4$

EVALUATION

Teachers are encouraged to develop their own formative (ongoing) evaluation strategies to monitor student achievement throughout the year. These strategies should include written and oral testing, small group work assignments, and whole class projects. Examples presented in the suggested instructional strategies sections of the *Syllabus* may be used as models for examination questions or assignments.

Although there are no plans at this time to develop a State final examination for this *Syllabus*, there is discussion of developing a norm-referenced Mathematics 7-8 Test to reflect the new *Syllabi*. Such a test could be used for program evaluation and/or student placement purposes.

This *Syllabus* for mathematics in grade 8 is designed to prepare students for either of two high school sequences. Those students pursuing a Regents diploma must pass Regents examinations in both Course I and Course II of the Three-Year Sequence for High School Mathematics. Those students pursuing a local diploma are required to pass either a Regents examination in mathematics, or the Regents Competency Test in Mathematics.

Students with handicapping conditions are to be provided with appropriate alternative testing techniques as indicated on their Individualized Education Programs (IEP). Such approved testing modifications should be used consistently in ongoing evaluations and final evaluations required for local and Regents diplomas.

APPENDIX A

GRADE 7: CONTENT OUTLINE AND SUGGESTED TIME ALLOTMENTS

Unit Topics	Time (Days)
I. An Introduction to Problem Solving	4-5
A. General Technique	
1. Understand what is happening in the problem	
2. Formulate a plan for solution	
3. Carry out the plan and observe the consequences	
4. Check the answer to see whether it makes sense	
B. Various Problem-Solving Strategies	
1. Use of pictures and diagrams	
2. Use of a simpler but related problem	
3. Patterns	
4. Trial and error (guessing)	
5. Working backwards	
6. Insight	
7. Miscellaneous examples	
II. Number Concepts	45-49
A. Whole Numbers	
1. Definition of natural numbers or counting numbers	
2. Definition of whole numbers	
a. Reading and writing numerals	
b. Rounding whole numbers	
3. Operations with whole numbers	
a. Properties	
b. Inverse operations	
4. Sequences	
5. Factors and exponents	
6. Prime and composite numbers	
7. Multiples and divisors	
a. Greatest common factor	
b. Least common multiple	
c. Rules for divisibility	
8. Principal square roots that are whole numbers	
9. Order of operations	
B. Integers	
1. Definition	
2. Absolute value	

3. Addition of integers
 - a. Subtraction of integers
 - b. Problem solving
4. Multiplication of integers
 - a. Division of integers
 - b. Problem solving
5. Order of operations

C. Rational Numbers

1. Definition and subsets of the rational numbers
2. Equivalent fractions
 - a. Reducing to lowest terms
 - b. Increasing to higher terms
3. Operations and problem solving with positive common fractions
 - a. Four basic operations
 - b. Simplifying complex fractions
4. Converting common fractions to decimal fractions
 - a. Terminating decimals
 - b. Repeating decimals
5. Converting decimal fractions to common fractions
6. Operations and problem solving with decimals
 - a. Four basic operations
 - b. Estimation
 - c. Rounding decimals
7. Scientific notation for numbers larger than or equal to one
8. Inequalities
 - a. Comparisons of rational numbers
 - b. Number line graphs
 - c. Problem solving

III. Ratio, Proportion, and Percent

25-30

A. Ratio

1. Definition and forms of notation
2. Rates as ratios
3. Problem solving

B. Proportion

1. Definition and forms of notation
 - a. Cross-products rule
 - b. Means and extremes
2. Solving proportions
3. Problem solving

C. Percent

1. Definition
2. Decimal/fraction/percent equivalents
3. Working with percents
 - a. Finding a percent of a number
 - b. Finding what percent one number is of another
 - c. Finding a number when a percent of it is given
4. Problem solving

IV. Probability and Statistics

23-26

A. Probability

1. Empirical vs. theoretical probability
2. Empirical probability
 - a. Cumulative relative frequency
 - b. Biased objects
3. Theoretical probability
 - a. Sample space or outcome set
 - b. Event
 - c. Random selection
4. Range of probabilities
 - a. Certainty
 - b. Impossibility
 - c. Probabilities between zero and one
 - d. Sum of the probabilities
 - e. Probability of complement
5. Probability of two independent events
 - a. Definition of independent events
 - b. The counting principle (multiplication principle)
 - c. Sample spaces for two independent events
 - (1) Tree diagrams
 - (2) List of ordered pairs
 - (3) Graph of ordered pairs
 - d. Product of independent probabilities

B. Statistics

1. Definition
 - a. Steps in a statistical study
 - b. Sample
2. Organization of data
 - a. Range
 - b. Intervals of equal length
 - c. Frequency distribution
3. Histogram
4. Measures of central tendency
 - a. Mean
 - b. Median
 - c. Mode
 - d. Problem solving

V. Algebra and Coordinate Geometry

20-23

A. Algebraic Expressions and Sentences

1. Concepts through applications
 - a. Variables, constants and domains
 - b. Open sentence vs. algebraic expression
2. Translating verbal phrases into algebraic form
 - a. Forming algebraic expressions
 - b. Evaluating algebraic expressions

B. First Degree Equations and Inequalities in One Unknown

1. Forming open sentences
2. Solving open sentences that are equations
 - a. Replacement
 - b. Inverse operations
3. Solving open sentences that are inequalities

C. Functions and the Coordinate Plane

1. Concepts through applications (functions of the form $y = ax$)
 - a. Table of values
 - b. Sentence
 - c. Formula
 - d. Coordinate graph (first quadrant)
 - e. Predictions
2. Functions of the form $y = ax + b$ (optional)
3. The coordinate plane

VI. Computer Awareness

8-14

A. Introduction to Computers

1. What is a computer?
2. Components of a modern-day digital computer
 - a. Input
 - b. Control
 - c. Arithmetic and logic unit
 - d. Memory
 - e. Output
3. Organizing a plan for a computer solution
 - a. Flowchart
 - b. Program
4. Uses of computers today
 - a. School
 - b. Home
 - c. Business/Industry
 - d. Government
 - e. Media

B. Mathematical Operations and Functions

1. Operations using computer symbols
2. Evaluating expressions in computer format
 - a. Order of operations
 - b. Use of decimals in output
 - c. Use of scientific notation in output
3. Functions used with computers
 - a. Absolute value functions, $ABS(X)$
 - b. Greatest integer function, $INT(X)$
4. Capabilities and limitations

C. Computer Programming (Optional)

1. Determine computer output of given programs
2. Algorithms used in programming

3. Write original programs
4. Additional activities

VII. Geometry and Measurement

20-23

A. Geometry

1. Undefined terms; point, line, plane
2. Line segment, ray, and pairs of lines
3. Angles
 - a. Definition and symbolism
 - b. Degree measure and classification
 - c. Protractor
 - d. Pairs of angles: complementary, supplementary, adjacent, and vertical
4. Simple closed curves
5. Polygons
 - a. Definition and types of polygons
 - b. Triangles
 - (1) Classification by angles
 - (2) Classification by sides
 - c. Quadrilaterals
 - (1) Trapezoid
 - (2) Parallelogram
 - (3) Rectangle
 - (4) Rhombus
 - (5) Square
 - d. Problem solving
6. Circles

B. Metric System

1. Basic units of length, mass, and capacity
2. Prefixes and symbols
3. Equivalent measures within the metric system
4. Other measures: temperature, time, area, and volume

C. Customary System of Measure

D. Measurement of Geometric Figures

1. Perimeter
 - a. Polygons
 - b. Circles: circumference
2. Area
 - a. Quadrilaterals
 - (1) Rectangle
 - (2) Square
 - (3) Parallelogram
 - (4) Rhombus
 - (5) Trapezoid (*optional*)
 - b. Triangles
 - c. Circles
3. Volume
 - a. Rectangular prism
 - b. Cube

APPENDIX B

STATEMENT OF REGENTS GOALS FOR ELEMENTARY AND SECONDARY SCHOOL STUDENTS – 1984

1. Each student will master communication and computation skills as a foundation to:
 - 1.1 Think logically and creatively.
 - 1.2 Apply reasoning skills to issues and problems.
 - 1.3 Comprehend written, spoken and visual presentations in various media.
 - 1.4 Speak, listen to, read, and write clearly and effectively in English.
 - 1.5 Perform basic mathematical calculations.
 - 1.6 Speak, listen to, read, and write at least one language other than English.
 - 1.7 Use current and developing technologies for academic and occupational pursuits.
 - 1.8 Determine what information is needed for particular purposes and be able to acquire, organize and use that information for those purposes.
2. Each student will learn methods of inquiry and knowledge gained through the following disciplines and use the methods and knowledge in interdisciplinary applications:
 - 2.1 English language and literature.
 - 2.2 History and social science.
 - 2.3 Mathematics.
 - 2.4 Natural sciences and technology.
 - 2.5 Language and literature in at least one language other than English.
3. Each student will acquire knowledge, understanding, and appreciation of the artistic, cultural, and intellectual accomplishments of civilization and develop the skills to express personal artistic talents. Areas include:
 - 3.1 Ways to develop knowledge and appreciation of the arts.
 - 3.2 Aesthetic judgments and the ability to apply them to works of art.
 - 3.3 Ability to use cultural resources of museums, libraries, theaters, historic sites, and performing arts groups.
 - 3.4 Ability to produce or perform works in at least one major art form.
 - 3.5 Materials, media and history of major art forms.
 - 3.6 Understanding of the diversity of cultural heritages.
4. Each student will acquire knowledge about political, economic, and social institutions and procedures in this country and other countries. Included are:
 - 4.1 Knowledge of American political, economic and social processes and policies at national, state, and local levels.
 - 4.2 Knowledge of political, economic, and social institutions and procedures in various nations; ability to compare the operation of such institutions; and understanding of the international interdependence of political, economic, social, cultural and environmental systems.

5. Each student will respect and practice basic civic values and acquire the skills, knowledge, understanding, and attitudes necessary to participate in democratic self-government. Included are:
 - 5.1 Understanding and acceptance of the values of justice, honesty, self-discipline, due process, equality, and majority rule with respect for minority rights.
 - 5.2 Respect for self, others and property as integral to a self-governing, democratic society.
 - 5.3 Ability to apply reasoning skills and the process of democratic government to resolve societal problems and disputes.
6. Each student will develop the ability to understand and respect people of different race; sex; ability; cultural heritage; national origin; religion; and political, economic, and social background, and their values, beliefs and attitudes.
7. Each student will acquire knowledge of the ecological consequences of choices in the use of the environment and natural resources.
8. Each student will develop general career skills, attitudes, and work habits and make a self-assessment of career prospects. Students not directly pursuing postsecondary education will acquire entry-level employment skills.
9. Each student will learn knowledge, skills, and attitudes which enable development of:
 - 9.1 Self-esteem.
 - 9.2 The ability to maintain physical, mental and emotional health.
 - 9.3 Understanding of the ill effects of alcohol, tobacco and other drugs.
10. Each student will develop a commitment to lifetime learning with the capacity for undertaking new studies, synthesizing new knowledge and experience with the known, and refining the ability to judge.

APPENDIX C

STUDENTS WITH HANDICAPPING CONDITIONS

The Board of Regents, through the Part 100 Regulations of the Commissioner and the Action Plan, has made a strong commitment to integrating the education of students with handicapping conditions into the total school program. According to Section 100.2(s) "Each student with a handicapping condition, as such term is defined in Section 200.1(ii) of this Chapter, shall have access to the full range of programs and services set forth in this Part to the extent that such programs and services are appropriate to such student's special educational needs." Districts must have policies and procedures in place to make sure that students with handicapping conditions have equal opportunities to access diploma credits, courses, and requirements.

The majority of students with disabilities have the potential to master the curricula content requirements for a high school diploma. Most students who require special education attend regular education classes in conjunction with specialized instruction and/or related services. These students must attain the same academic standards as their nonhandicapped peers in order to meet these requirements. For this reason, it is very important that at all grade levels students with handicapping conditions receive instruction in the same content areas so as to receive the same informational base that will be required for proficiency on statewide testing programs and diploma requirements.

It is important that teachers providing instruction through this syllabus/curriculum be familiar with how the process of special education is implemented. This appendix will provide clarification as to how students with handicapping conditions participate in both regular and special education programs. The coordination and integration of both of these programs is essential for the benefit of all students.

Providing Special Education

A student who is suspected of having a handicapping condition which adversely affects the student's educational performance must be referred to the Committee on Special Education (CSE) by either the student's parent, a professional staff member of the school, a physician, a judicial officer, the commissioner or designee of a public agency with responsibility for welfare, health or education of children, or the student, on his or her own behalf, if he or she is 18 years of age or an emancipated minor. The referral must be in writing.

The CSE then must proceed to conduct an individual evaluation. This evaluation is a comprehensive assessment which is used by the CSE to determine whether the student has a handicapping condition which adversely affects the student's educational performance. If the Committee on Special Education determines that the student is eligible for special education, the CSE and the student's parents develop an individualized education program (IEP). This recommendation must include:

- present levels of performance and individual needs according to the four criteria areas of academic, social, and physical development and management needs,
- classification of the handicapping condition,
- annual goals,
- recommended program, class size, extent of participation in regular education,
- projected date for initiation of special education, amount of time per day and projected review date,
- specialized equipment and adaptive devices,

- testing modifications, and
- recommended placement.

Following the CSE recommendation, the Board of Education (BOE) has 30 school days to arrange for implementation. After the student enters the program, a planning conference is held to complete the development of the individualized education program (IEP) by adding short-term objectives and objective criteria and evaluation procedures for determining whether the instructional objectives are being met. The student's program is reviewed at least annually and the student receives a comprehensive reevaluation at least every three years.

The entire process has corresponding timelines and due process procedures for each step.

The teacher providing instruction through this syllabus/ curriculum has the opportunity to provide an educational setting which will enable the students to explore their abilities and interests. Instruction could be provided to handicapped students either by teachers certified in this subject area or by special education teachers. Teachers certified in this subject area would be providing instruction to handicapped students recommended by the CSE as being able to benefit from instruction in a regular educational setting. Special education teachers may also provide this instruction to a class of students with handicapping conditions in a special class setting.

Teachers certified in the subject area should become aware of the needs of those students with handicapping conditions participating in their classes. Instructional techniques and materials must be modified to the extent appropriate to provide students with handicapping conditions the opportunity to meet diploma requirements. Information or assistance is available through special education teachers, administrators, the Committee on Special Education or a student's IEP.

Special education teachers providing this instruction must also become familiar with the goals and objectives of the curriculum. It is important that these teachers provide their students with the same or equivalent information contained in the curriculum.

Regardless of who provides the instruction, the cooperation between teachers of regular and special education programs is essential. It is important for the students as well as the total school environment.

Alternative Testing Techniques

Another consideration in assisting students with handicapping conditions to meet the requirements of regular education is the use of alternative testing techniques. Alternative testing techniques are modifications of testing procedures or formats which provide students with handicapping conditions equal opportunity to participate in testing situations. Such techniques provide the opportunity to demonstrate mastery of skills and attainment of knowledge without being limited or unfairly restricted by the existence of a handicapping condition.

The Committee on Special Education (CSE) is responsible for identifying and documenting the student's need for alternative testing techniques. This determination is made when a student is initially referred to the CSE, is reviewed annually for as long as the student receives special education services, and is reviewed when the student is determined to no longer need special education services. **These modifications are to be used consistently throughout the student's educational program.** Principals ensure that students who have been identified by the CSE as educationally handicapped are provided the alternative testing techniques which have been recommended by the CSE and approved by the Board of Education.

Alternative testing techniques which have been approved for use by a student must be used consistently in both special and regular education settings. Regular classroom teachers should be aware of possible alternative testing techniques and should be skilled in their implementation.

The coordination and cooperation of the total school program will assist in providing the opportunity for a greater number of handicapped students to meet the requirements needed to pursue a high school diploma. The integrated provision of regular education programs, special education programs, remediation, alternative testing techniques, modified teacher techniques

and materials, and access to credit through alternatives will assist in enabling such students to pursue the high school diploma to a greater degree. The teacher who provides instruction through this curriculum has a unique opportunity to assist such students in their individual goals.

Infusing Awareness of Persons with Disabilities in Curriculum

Throughout the development of the Action Plan, the Board of Regents reaffirmed a strong commitment to integrating the education of students with handicapping conditions into the total school program. In keeping with the concept of integration, the following subgoal of the Action Plan was established:

In all subject areas, revisions in the syllabi will include materials and activities related to generic subgoals such as problem solving, reasoning skills, speaking, capacity to search for information, the use of libraries and increasing student awareness of and information about the disabled.

The purpose of this subgoal is to ensure that appropriate activities and materials are available to increase student awareness of disabilities and issues in regard to disabilities.

This syllabus/curriculum, by design, includes information, activities and materials regarding persons with handicapping conditions. Teachers are encouraged to include other examples as may be appropriate to their classroom or the situation at hand.